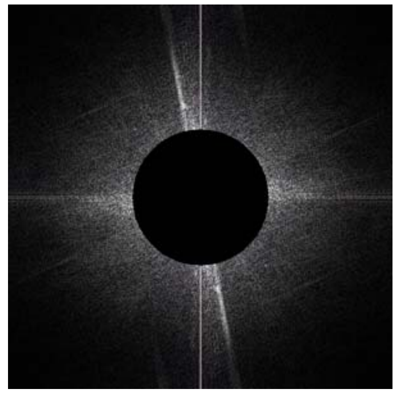
Filters



Basic Signal Processing

Topics

Filters in the spatial domain

Spectral representation

Fourier transforms

The convolution theorem

Filters in the frequency domain

Next week:

- Sampling, aliasing and antialiasing
- Compression

Filter = Convolution

Convolution

Signal/Image

1	3	0	4	2	1	...

Filter

1	2

Convolution

1	3	0	4	2	1	…
1	2					

1 * 1 + 2 * 3 = 7

7						…

Convolution

1	3	0	4	2	1	…
	1	2				

1 * 3 + 2 * 0 = 3

7	3					…

Convolution

1	3	0	4	2	1	…
		1	2			

1 * 0 + 2 * 4 = 8

7	3	8				…

Convolution of a “Spike”

Signal/Image

0	0	1	0	0	0	…

Filter

1	2	1

Result: copy of the filter centered at the spike

0	1	2	1	0	0	…

Convolution of a Two Boxes

Signal/Image

0	0	1	1	0	0	…

Filter

1	1

Result: Convolution of two boxes is a triangle

0	1	2	1	0	0	…

Mathematical Definition: Convolution

Signal x_i

Filter h_i

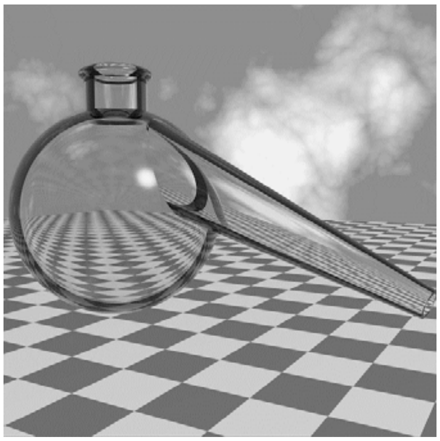
Convolution

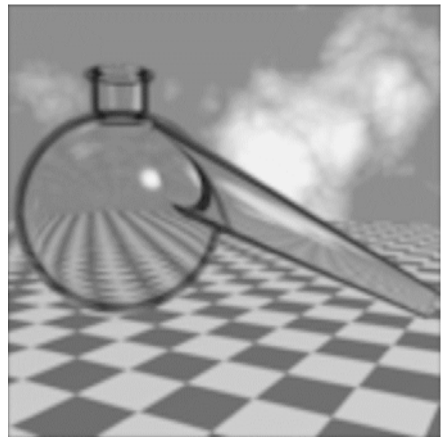
$$y = h \otimes x$$

$$y_i = \sum_j x_j h_{i-j}$$

Low-Pass Filter

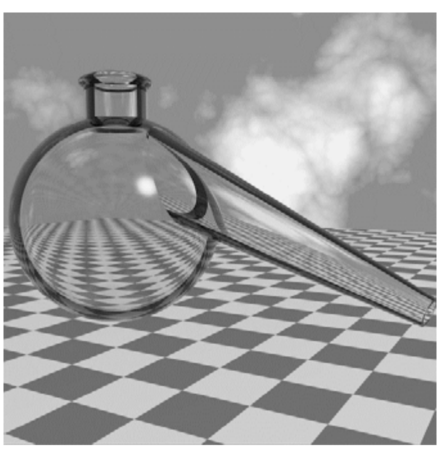
Original | **Blurred**

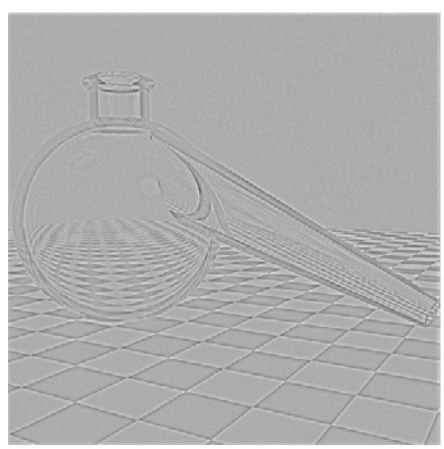




High-Pass Filter

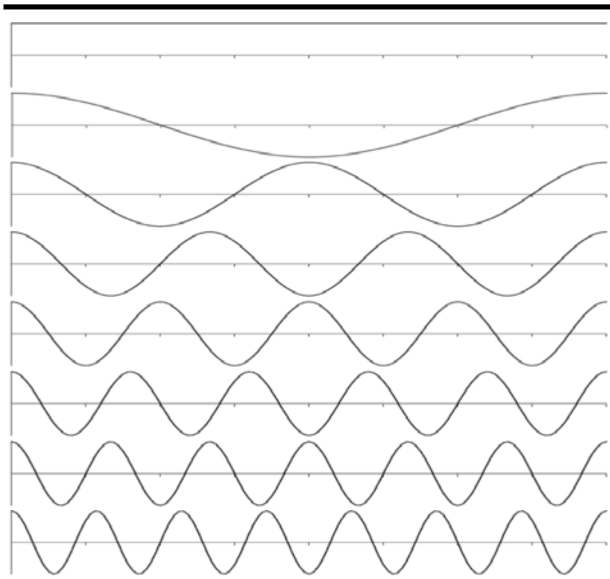
Original | **Edge Enhancement**





Basic Signal Processing

Sines and Cosines



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Frequency

Frequency (cycles per interval) $f = \frac{1}{T}$

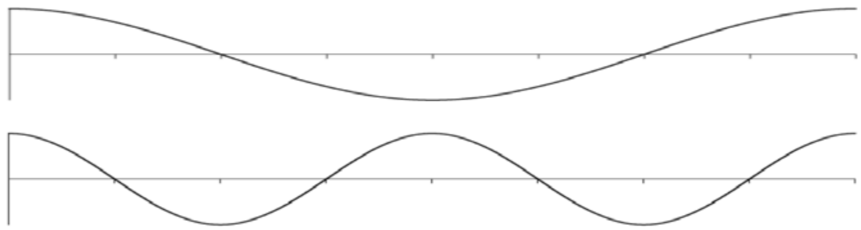
Angular frequency $\omega = 2\pi f = \frac{2\pi}{T}$

Spectral Representation

Functions may be represented as a sum of sin/cos

$$f(x) = \sum_i a_i \cos \frac{2\pi i x}{T} + b_i \sin \frac{2\pi i x}{T}$$

Cosines are Orthogonal



$$\int \cos \omega_i x \cos \omega_j x \, dx = \begin{cases} 1 & \omega_i = \omega_j \\ 0 & \omega_i \neq \omega_j \end{cases}$$

Fourier Coefficients

$$a_i = \int f(x) \cos(\frac{2\pi i}{T} x) \, dx$$

$$b_i = \int f(x) \sin(\frac{2\pi i}{T} x) \, dx$$

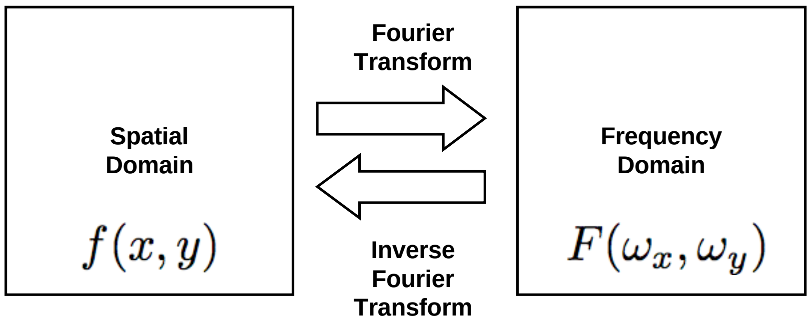
Power spectra: $F(\omega_i) = \sqrt{a_i^2 + b_i^2}$

Two Domains

Two representations of a function

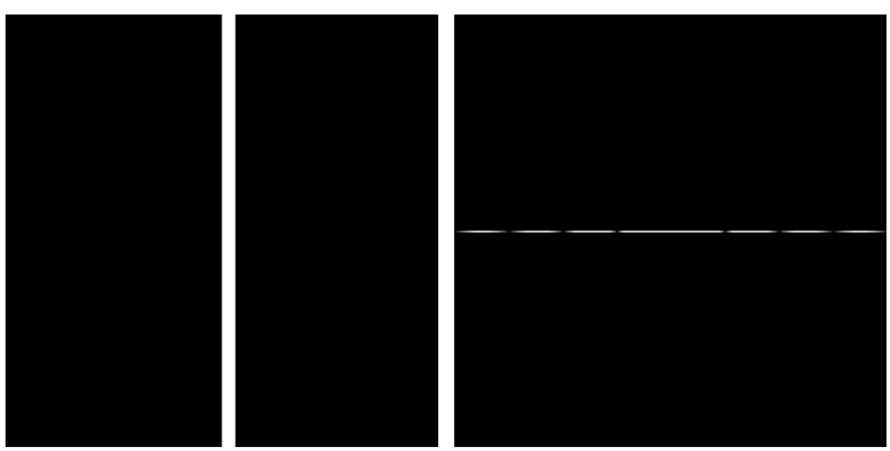
- Spatial domain - normal representation
- Frequency domain - spectral representation

The *Fourier transform* converts between domains



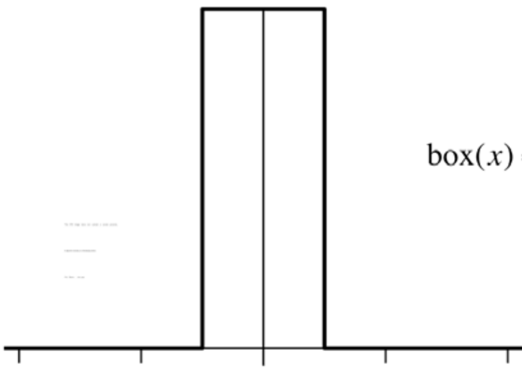
Box Function and Sinc Function

Spatial Domain | **Frequency Domain**



$$f(x) \Leftrightarrow F(\omega_x)$$

The Box Function



$$\mathrm{box}(x) = \begin{cases} 0 & x < -1/2 \\ 1 & -1/2 < x < 1/2 \\ 0 & x > 1/2 \end{cases}$$

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The “Sinc” Function

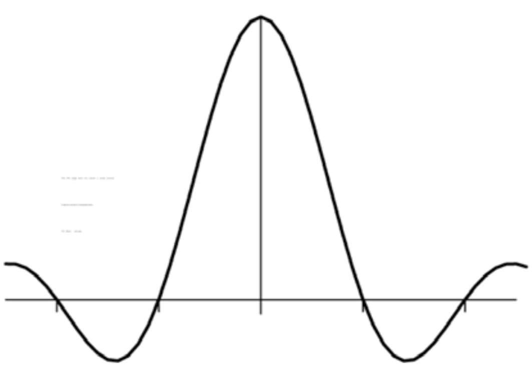
$$\int_{-\infty}^{\infty} \mathrm{box}(\mathrm{x}) \cos 2\pi f x \, \mathrm{d}x = \int_{-\frac{1}{2}}^{-\frac{1}{2}} \cos 2\pi f x \, \mathrm{d}x$$

$$= \left. \frac{\sin 2\pi f x}{2\pi f} \right|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{\sin \frac{1}{2} 2\pi f - \sin - \frac{1}{2} 2\pi f}{2\pi f}$$

$$= \frac{\sin \pi f + \sin \pi f}{2\pi f}$$

$$= \frac{\sin \pi f}{\pi f}$$



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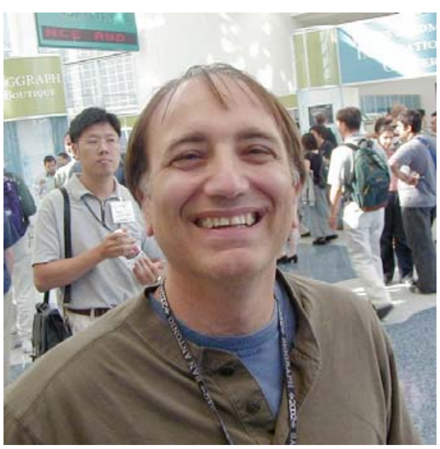
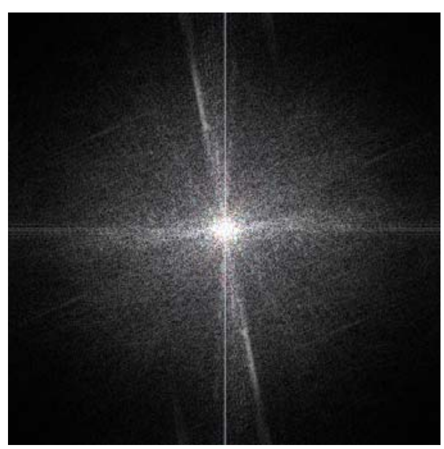
Gallery of Properties

Pat’s Frequencies

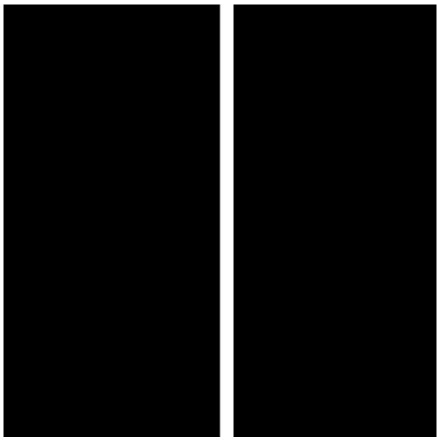
Spatial Domain | **Frequency Domain**

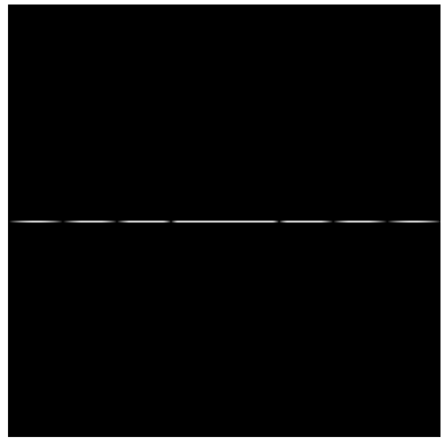




Product Property

Spatial Domain | **Frequency Domain**





$$f(x) \Leftrightarrow F(\omega_x)$$

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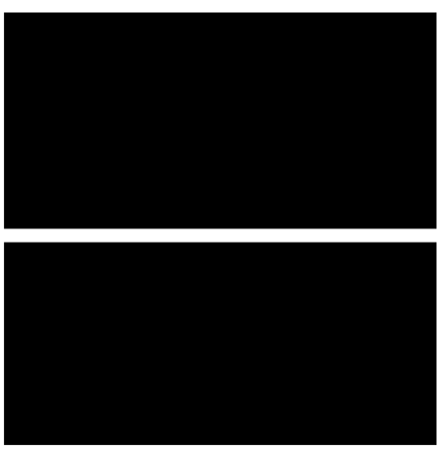
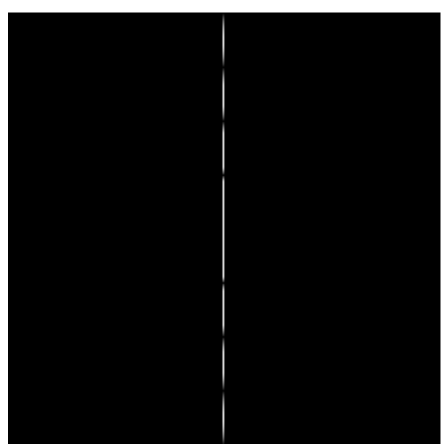
Product Property

Spatial Domain | **Frequency Domain**





$$f(y) \Leftrightarrow F(\omega_y)$$

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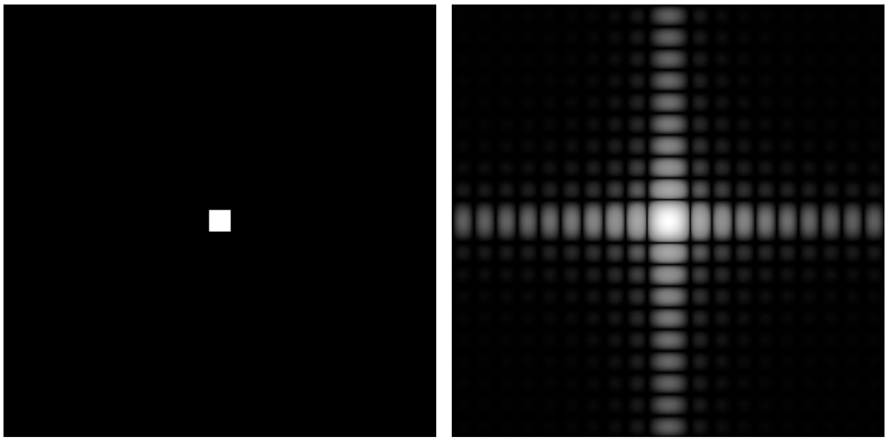
Product Property

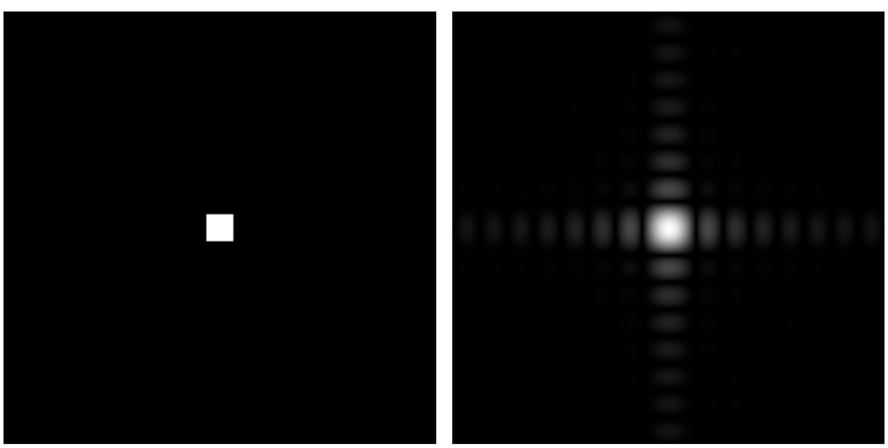
Spatial Domain | **Frequency Domain**



$$f(x)f(y) \Leftrightarrow F(\omega_x, \omega_y)$$

Scaling Property

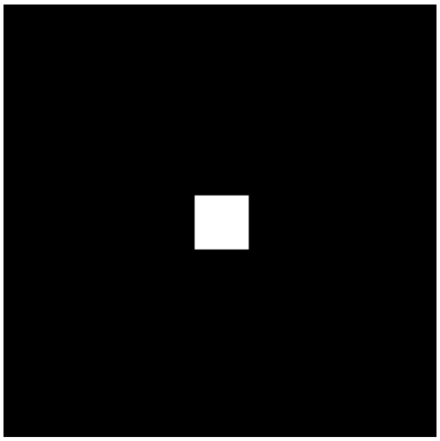
Spatial Domain | **Frequency Domain**

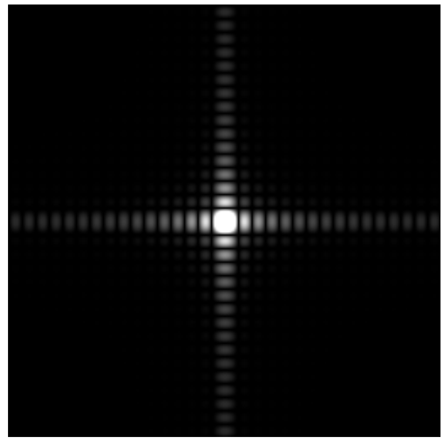


$$f(ax) \Leftrightarrow F(\omega/a)$$

Scaling Property

Spatial Domain | **Frequency Domain**

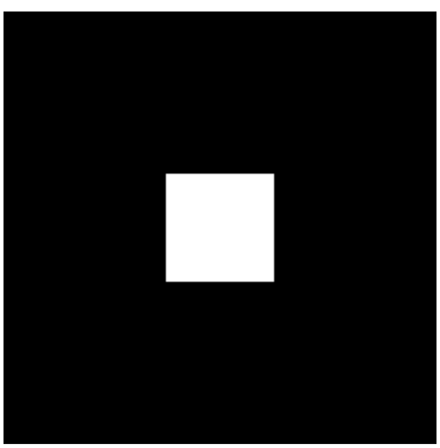


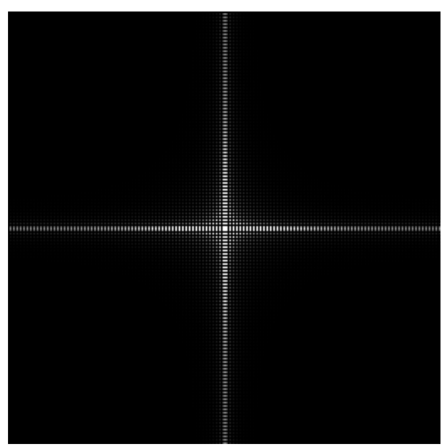


$$f(ax) \Leftrightarrow F(\omega/a)$$

Scaling Property

Spatial Domain | **Frequency Domain**





$$f(ax) \Leftrightarrow F(\omega/a)$$

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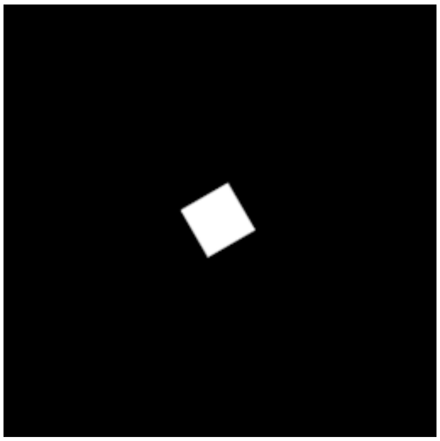
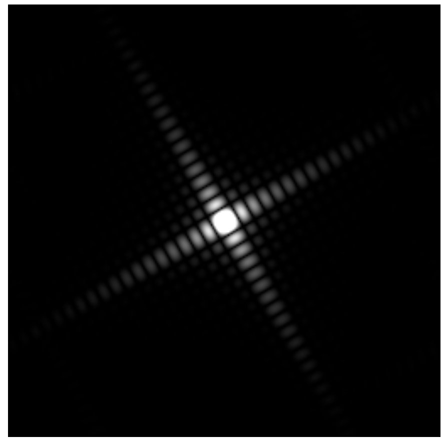
Rotation Property

Spatial Domain | **Frequency Domain**





$$f(Rx) \Leftrightarrow F(R\omega)$$

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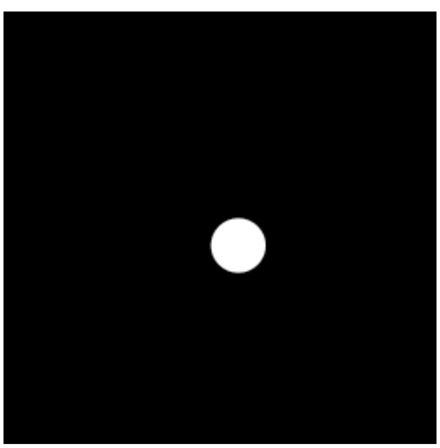
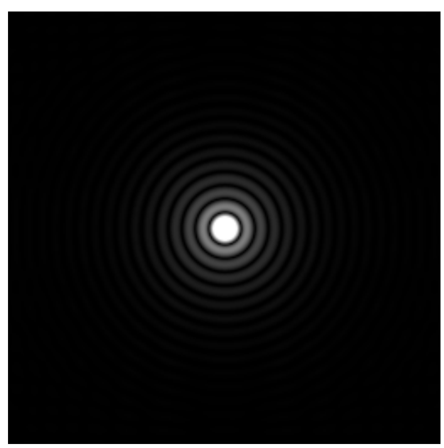
Rotation Property

Spatial Domain | **Frequency Domain**





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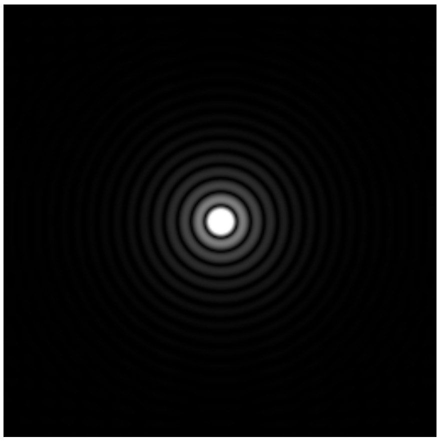
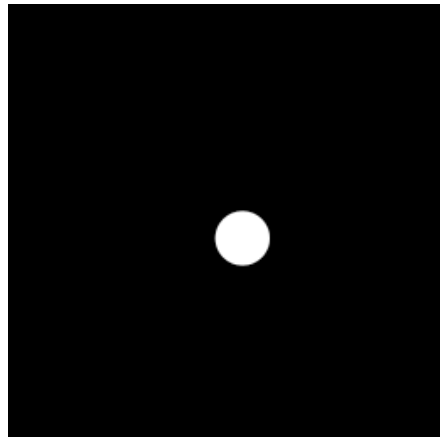
Symmetry Property

Spatial Domain | **Frequency Domain**





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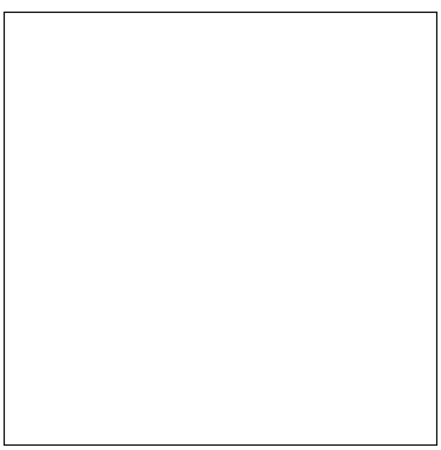
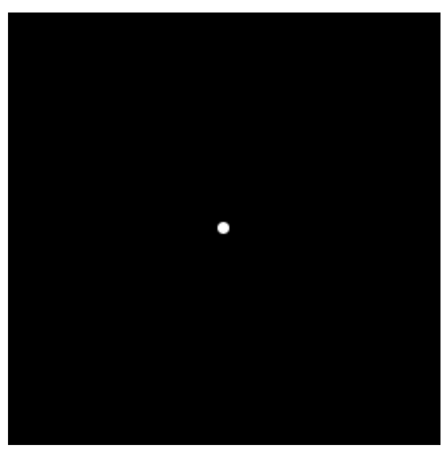
Symmetry Property

Spatial Domain | **Frequency Domain**





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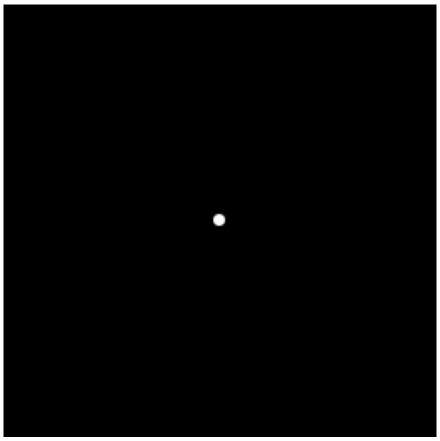
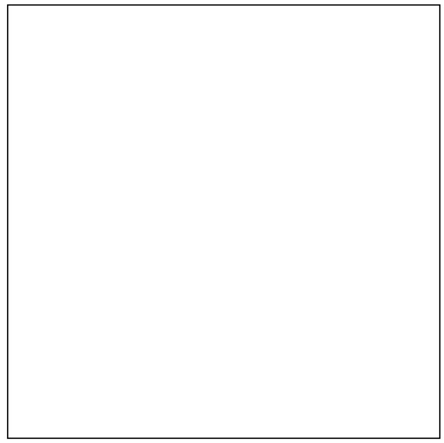
Symmetry Property

Spatial Domain | **Frequency Domain**





Convolution Theorem

Convolution

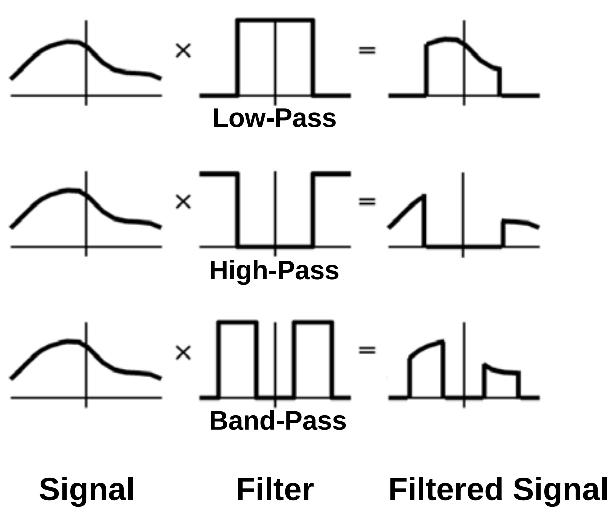
Definition:

$$h(x) = f \otimes g = \int f(x')g(x - x')\,dx'$$

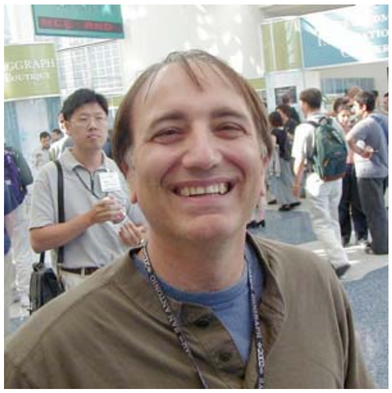
***Convolution Theorem*: Multiplication in the frequency domain is equivalent to convolution in the space domain.**

$$f \otimes g \leftrightarrow F \times G$$

Filters in Frequency Space



Fourier Transform of Pat



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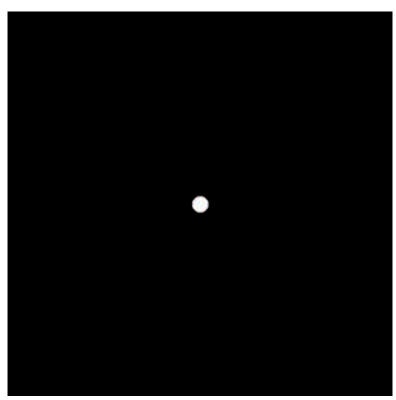
Low-Pass Pat



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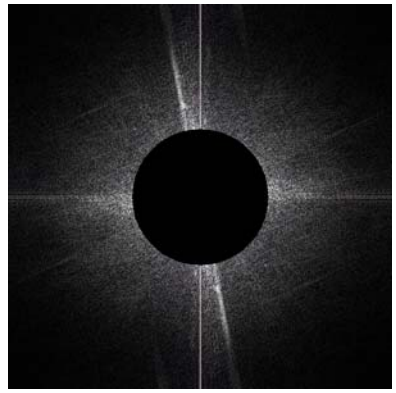
High-Pass Pat



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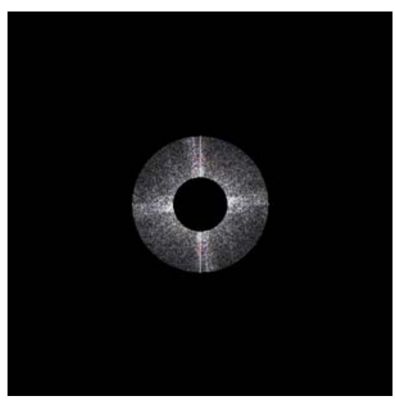
Band-Pass Pat



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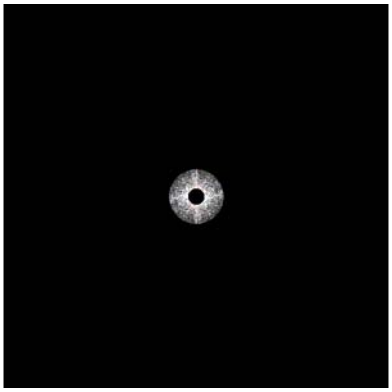
Band-Pass Pat



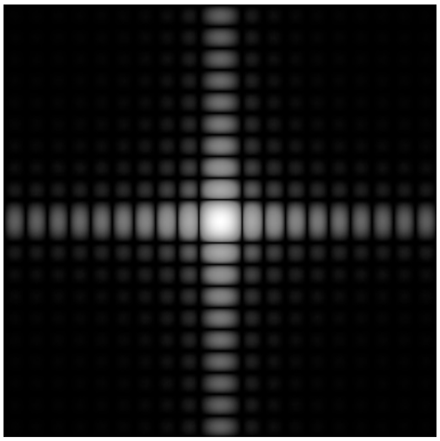
Unsharp Masking

Subtract blurred image from original to sharpen

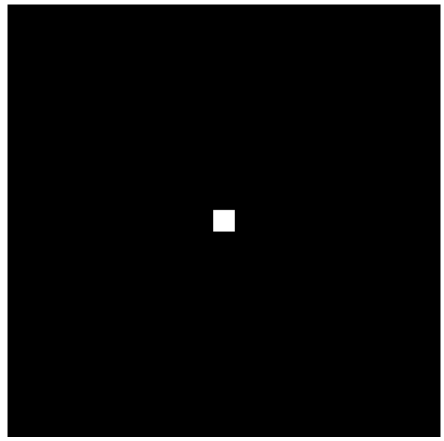
Original - Blurred = Sharpened

Perfect Low-Pass = Sinc Convolution

Spatial Domain



Frequency Domain



Convolution

***Convolution Theorem*: Multiplication in the frequency domain is equivalent to convolution in the space domain.**

$$f \otimes g \leftrightarrow F \times G$$

***Symmetric Theorem*: Multiplication in the space domain is equivalent to convolution in the frequency domain.**

$$f \times g \leftrightarrow F \otimes G$$

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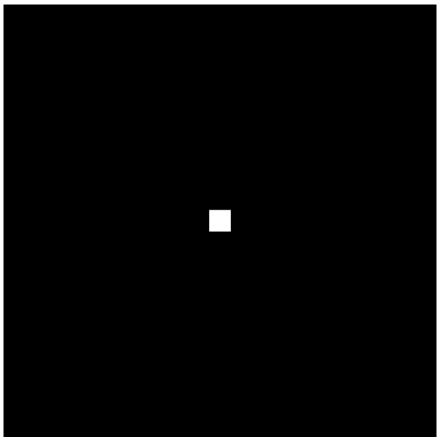
Box Convolution = Sinc-Pass Filter

Spatial Domain



Frequency Domain