# Welcome!

# MiniSymposium on Algorithms and Software to Compute Conservation Laws of Nonlinear PDEs

Speakers: Hereman, Bluman, Nivala, Wolf

# Symbolic Computation of Conservation Laws of Nonlinear PDEs in Multi-dimensions


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Minisymposium MS29
2006 SIAM Conference on Nonlinear Waves
and Coherent Structures
University of Washington, Seattle, WA

Sunday, September 10, 2006, 3:30p.m.

# Acknowledgements

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Graduate students Douglas Poole (CSM), Michael Nivala (UW).

Several undergraduate and graduate students.

Research supported in part by NSF
under Grant CCR-9901929


This presentation was made in `TeXpower`

# Outline

- Conservation Laws of PDEs in multi-dimensions
- Example: shallow water wave equations (Dellar)
- Algorithmic Methods for conservation laws
- Computer Demonstration
- Tools:
  - Euler operators (testing exactness)
  - Calculus-based formulas for homotopy operator
    - ⋆ symbolic integration by parts
    - ⋆ inversion of the total divergence operator
- Application: shallow water wave equations
- Conclusions: future work, software and publications

# Notations – Computations on the Jet Space

- Independent variables $\mathbf{x} = (x, y, z)$
- Dependent variables $\mathbf{u} = (u^{(1)}, u^{(2)}, \ldots, u^{(j)}, \ldots, u^{(N)})$
  In examples: $\mathbf{u} = (u, v, \theta, h, \ldots)$
- Partial derivatives $u_{kx} = \frac{\partial^k u}{\partial x^k}$, $u_{kx\,ly} = \frac{\partial^{k+l} u}{\partial x^k y^l}$, etc.
- *Differential functions*
  Example: $f = uvv_x + x^2 u_x^3 v_x + u_x v_{xx}$ for $u(x)$, $v(x)$
- Total derivative (with respect to $x$)

$$\mathrm{D}_x = \frac{\partial}{\partial x} + \sum_{k=0}^{M_x^{(1)}} u_{(k+1)x} \frac{\partial}{\partial u_{kx}} + \sum_{k=0}^{M_x^{(2)}} v_{(k+1)x} \frac{\partial}{\partial v_{kx}}$$

$M_x^{(1)}$ is the order of $f$ in $u$ (with respect to $x$), $etc.$

# Conservation Laws

- Conservation law in $(1+1)$ dimensions

$$\boxed{\mathrm{D}_t \rho + \mathrm{D}_x J = 0} \quad \text{(on PDE)}$$

  conserved density $\rho$ and flux $J$

- Example: Korteweg-de Vries (KdV) equation

  $u_t + u u_x + u_{3x} = 0$

- Sample conservation law

$$\mathrm{D}_t \left( u^3 - 3{u_x}^2 \right) +$$

$$\mathrm{D}_x \left( \frac{3}{4} u^4 - 6 u {u_x}^2 + 3 u^2 u_{2x} + 3 {u_{2x}}^2 - 6 u_x u_{3x} \right) = 0$$

- Key property: Dilation invariance
- Example: KdV equation and its density-flux pairs are invariant under the scaling symmetry

$$(x, t, u) \rightarrow (\frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u),$$

$\lambda$ is arbitrary parameter.

- Some density-flux pairs for the KdV equation:

$$\rho^{(1)} = u \qquad J^{(1)} = \frac{u^2}{2} + u_{2x}$$

$$\rho^{(2)} = u^2 \qquad J^{(2)} = \frac{2u^3}{3} + 2uu_{2x} - {u_x}^2$$

$$\rho^{(3)} = u^3 - 3{u_x}^2$$

$$J^{(3)} = \frac{3}{4}u^4 - 6u{u_x}^2 + 3u^2u_{2x} + 3{u_{2x}}^2 - 6u_xu_{3x}$$

$$\vdots$$

$$\rho^{(6)} = u^6 - 60\,u^3{u_x}^2 - 30\,{u_x}^4 + 108\,u^2{u_{2x}}^2 + \frac{720}{7}\,{u_{2x}}^3 - \frac{648}{7}\,u{u_{3x}}^2 + \frac{216}{7}\,{u_{4x}}^2$$

$$\vdots$$

- Conservation law in $(3+1)$ dimensions

$$\boxed{\mathrm{D}_t\rho + \nabla\cdot\mathbf{J} = \mathrm{D}_t\rho + \mathrm{D}_x J_1 + \mathrm{D}_y J_2 + \mathrm{D}_z J_3 = 0} \text{ (on PDE)}$$

conserved density $\rho$ and flux $\mathbf{J} = (J_1, J_2, J_3)$

- Example: Shallow water wave (SWW) equations

[P. Dellar, Phys. Fluids **15** (2003) 292-297]

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + 2\,\mathbf{\Omega}\times\mathbf{u} + \nabla(\theta h) - \frac{1}{2}h\nabla\theta = \mathbf{0}$$

$$\theta_t + \mathbf{u}\cdot(\nabla\theta) = 0$$

$$h_t + \nabla\cdot(\mathbf{u}h) = 0$$

where $\mathbf{u}(x,y,t), \theta(x,y,t)$ and $h(x,y,t)$.

- In components:

$$u_t + uu_x + vu_y - 2\,\Omega v + \frac{1}{2}h\theta_x + \theta h_x = 0$$
$$v_t + uv_x + vv_y + 2\,\Omega u + \frac{1}{2}h\theta_y + \theta h_y = 0$$
$$\theta_t + u\theta_x + v\theta_y = 0$$
$$h_t + hu_x + uh_x + hv_y + vh_y = 0$$

- SWW equations are invariant under

$$(x, y, t, u, v, h, \theta, \Omega) \to$$
$$(\lambda^{-1}x, \lambda^{-1}y, \lambda^{-b}t, \lambda^{b-1}u, \lambda^{b-1}v, \lambda^{a}h, \lambda^{2b-a-2}\theta, \lambda^{b}\Omega)$$

where $W(h) = a$ and $W(\Omega) = b \quad (a, b \in \mathbb{Q})$.

- First few densities-flux pairs of SWW system:

$$\rho^{(1)} = h \qquad \mathbf{J}^{(1)} = \begin{pmatrix} uh \\ vh \end{pmatrix}$$

$$\rho^{(2)} = h\theta \qquad \mathbf{J}^{(2)} = \begin{pmatrix} uh\theta \\ vh\theta \end{pmatrix}$$

$$\rho^{(3)} = h\theta^2 \qquad \mathbf{J}^{(3)} = \begin{pmatrix} uh\theta^2 \\ vh\theta^2 \end{pmatrix}$$

$$\rho^{(4)} = (u^2+v^2)h + h^2\theta \qquad \mathbf{J}^{(4)} = \begin{pmatrix} u^3h + uv^2h + 2uh^2\theta \\ v^3h + u^2vh + 2vh^2\theta \end{pmatrix}$$

$$\rho^{(5)} = v_x\theta - u_y\theta + 2\Omega\theta$$

$$\mathbf{J}^{(5)} = \frac{1}{2}\begin{pmatrix} 4\Omega u\theta - 2uu_y\theta + 2uv_x\theta - h\theta\theta_y \\ 4\Omega v\theta + 2vv_x\theta - 2vu_y\theta + h\theta\theta_x \end{pmatrix}$$

# Computer Demonstration

# Algorithmic Methods for Conservation Laws

- Use Noether's Theorem (Lagrangian formulation).
- Direct methods (Anderson, Bluman, Anco, Wolf, etc.) based on solving ODEs (or PDEs).
- Strategy (linear algebra and variational calculus).
  - Density is linear combination of scaling invariant terms with undetermined coefficients.
  - Use variational derivative (Euler operator) to compute the undetermined coefficients.
  - Use the homotopy operator to compute the flux (invert $\mathrm{D}_x$ or $\mathrm{Div}$) (Deconinck and Nivala).
  - Work with linearly independent pieces in finite dimensional spaces.

# Review of Vector Calculus

- The curl annihilates gradients!
- The divergence annihilates curls!
- The Euler operator annihilates divergences!

Formula for Euler operator (variational derivative)

in 1D:

$$\begin{aligned}\mathcal{L}^{(0)}_{u^{(j)}(x)} &= \sum_{k=0}^{M_x^{(j)}} (-\mathrm{D}_x)^k \frac{\partial}{\partial u^{(j)}_{kx}} \\ &= \frac{\partial}{\partial u^{(j)}} - \mathrm{D}_x \frac{\partial}{\partial u^{(j)}_x} + \mathrm{D}_x^2 \frac{\partial}{\partial u^{(j)}_{2x}} - \mathrm{D}_x^3 \frac{\partial}{\partial u^{(j)}_{3x}} + \cdots\end{aligned}$$

# Formula for Euler operator in 2D:

$$\mathcal{L}^{(0,0)}_{u^{(j)}(x,y)} = \sum_{k_x=0}^{M_x^{(j)}} \sum_{k_y=0}^{M_y^{(j)}} (-\mathrm{D}_x)^{k_x} (-\mathrm{D}_y)^{k_y} \frac{\partial}{\partial u^{(j)}_{k_x x\, k_y y}}$$

$$= \frac{\partial}{\partial u^{(j)}} - \mathrm{D}_x \frac{\partial}{\partial u^{(j)}_x} - \mathrm{D}_y \frac{\partial}{\partial u^{(j)}_y}$$

$$+ \mathrm{D}_x^2 \frac{\partial}{\partial u^{(j)}_{2x}} + \mathrm{D}_x \mathrm{D}_y \frac{\partial}{\partial u^{(j)}_{xy}} + \mathrm{D}_y^2 \frac{\partial}{\partial u^{(j)}_{2y}} - \mathrm{D}_x^3 \frac{\partial}{\partial u^{(j)}_{3x}} \cdots$$

# Inverting $\mathrm{D}_x$ and $\mathrm{Div}$

## Problem Statement

- **In 1D:**

  Example: For $u(x)$ and $v(x)$

  $f = 3u_x v^2 \sin u - u_x^3 \sin u - 6 v v_x \cos u + 2 u_x u_{2x} \cos u + 8 v_x v_{2x}$

- Find $F = \int f \, dx$ Thus, $f = D_x F$.

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- Find $F = \int f \, dx$ Thus, $f = D_x F$.
- Result (by hand):

$$F = 4\, v_x^2 + u_x^2 \, \cos u - 3\, v^2 \, \cos u$$

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**Mathematica cannot compute this integral!**

- **In 2D or 3D:**

Example: For $u(x,y)$ and $v(x,y)$

$$f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$$

- Find $\mathbf{F} = \mathrm{Div}^{-1} f$ Thus, $f = \mathrm{Div}\,\mathbf{F}$.

- **In 2D or 3D:**

  Example: For $u(x, y)$ and $v(x, y)$

$$f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$$

- Find $\mathbf{F} = \mathrm{Div}^{-1} f$ Thus, $f = \mathrm{Div}\,\mathbf{F}$.

- Result (by hand):

$$\tilde{\mathbf{F}} = (u v_y - u_x v_y, -u v_x + u_x v_x)$$

- **In 2D or 3D:**

  Example: For $u(x,y)$ and $v(x,y)$

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Can this be done without integration by parts?

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**Mathematica cannot do this!**

Can this be done without integration by parts?

Can this be reduced to single integral in one variable?

- **In 2D or 3D:**

  Example: For $u(x,y)$ and $v(x,y)$

$$f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$$

- Find $\mathbf{F} = \mathrm{Div}^{-1} f$ Thus, $f = \mathrm{Div}\,\mathbf{F}$.
- Result (by hand):

$$\tilde{\mathbf{F}} = (u v_y - u_x v_y, -u v_x + u_x v_x)$$

**Mathematica cannot do this!**

Can this be done without integration by parts?

Can this be reduced to single integral in one variable?

**Yes! With the Homotopy operator.**

# Integration by Parts with Homotopy Operator

- Theorem (integration with homotopy operator):

- In 1D: If $f$ is exact then

$$\boxed{F = \mathrm{D}_x^{-1} f = \int f \, dx = \mathcal{H}_{\mathbf{u}(x)} f}$$

- In 2D: If $f$ is a divergence then

$$\boxed{\mathbf{F} = \mathrm{Div}^{-1} \, f = (\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f, \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f)}$$

- In 3D: If $f$ is a divergence then

$$\boxed{\mathbf{F} = \mathrm{Div}^{-1} \, f = (\mathcal{H}^{(x)}_{\mathbf{u}(x,y,z)} f, \mathcal{H}^{(y)}_{\mathbf{u}(x,y,z)} f, \mathcal{H}^{(z)}_{\mathbf{u}(x,y,z)} f)}$$

## Simplified Formula for Homotopy Operator

- In 1D (with variable $x$):

$$\mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 \sum_{j=1}^{N} (I_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}$$

with integrand

$$I_{u^{(j)}} f = \sum_{i=0}^{M_x^{(j)}-1} \mathrm{D}_x^i \left( u^{(j)} \, \mathcal{L}_{u^{(j)}(x)}^{(i+1)} f \right)$$

$$= \sum_{i=0}^{M_x^{(j)}-1} u_{ix}^{(j)} \sum_{k=i+1}^{M_x^{(j)}} (-\mathcal{D}_x)^{k-(i+1)} \, \frac{\partial f}{\partial u_{kx}^{(j)}}$$

$N$ dependent variables, $(I_{u^{(j)}} f)[\lambda \mathbf{u}]$ means that in $I_{u^{(j)}} f$ one replaces $\mathbf{u}(x) \to \lambda \mathbf{u}(x)$, $\mathbf{u}_x(x) \to \lambda \mathbf{u}_x(x), \ldots$

- In 2D (with variables $x$ and $y$):

$$\boxed{\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f = \int_0^1 \sum_{j=1}^{N} (I^{(x)}_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}}$$

with integrand

$$\mathcal{I}^{(x)}_{u^{(j)}} f = \sum_{i_x=0}^{M_x^{(j)}-1} \sum_{i_y=0}^{M_y^{(j)}} \left( \frac{1+i_x}{1+i_x+i_y} \right) \mathrm{D}_x^{i_x} \mathrm{D}_y^{i_y} \left( u^{(j)} \mathcal{L}^{(1+i_x, i_y)}_{u^{(j)}(x,y)} f \right)$$

$$= \sum_{i_x=0}^{M_x^{(j)}-1} \sum_{i_y=0}^{M_y^{(j)}} \binom{i_x+i_y}{i_x} u_{i_x x \, i_y y} \sum_{k_x=i_x+1}^{M_x^{(j)}} \sum_{k_y=i_y}^{M_y^{(j)}} \frac{\binom{k_x+k_y-i_x-i_y-1}{k_x-i_x-1}}{\binom{k_x+k_y}{k_x}}$$

$$(-\mathcal{D}_x)^{k_x-i_x-1} \, (-\mathcal{D}_y)^{k_y-i_y} \, \frac{\partial f}{\partial u^{(j)}_{k_x x \, k_y y}}$$

Analogous formulas for $\mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f$ and $I^{(y)}_{u^{(j)}} f$

# Application of Homotopy Operator in 1D

Example:

$$f = 3u_x v^2 \sin u - u_x^3 \sin u - 6 v v_x \cos u + 2 u_x u_{2x} \cos u + 8 v_x v_{2x}$$

- Compute

$$\begin{aligned} I_u f &= u \frac{\partial f}{\partial u_x} + u_x \frac{\partial f}{\partial u_{2x}} - u \mathrm{D}_x \frac{\partial f}{\partial u_{2x}} \\ &= 3 u v^2 \sin u - u u_x^2 \sin u + 2 u_x^2 \cos u \end{aligned}$$

- Similarly,

$$\begin{aligned} I_v f &= v \frac{\partial f}{\partial v_x} + v_x \frac{\partial f}{\partial v_{2x}} - v \mathrm{D}_x \frac{\partial f}{\partial v_{2x}} \\ &= -6 v^2 \cos u + 8 v_x^2 \end{aligned}$$

- Finally,

$$\begin{aligned}
F &= \mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 \left(I_u f + I_v f\right)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \\
&= \int_0^1 \Big(3\lambda^2 u v^2 \sin(\lambda u) - \lambda^2 u u_x^2 \sin(\lambda u) + 2\lambda u_x^2 \cos(\lambda u) \\
&\qquad -6\lambda v^2 \cos(\lambda u) + 8\lambda v_x^2\Big)\, d\lambda \\
&= 4v_x^2 + u_x^2 \cos u - 3v^2 \cos u
\end{aligned}$$

# Computation of Conservation Laws for SWW

## Quick Recapitulation

- Conservation law in $(2+1)$ dimensions

$$\boxed{\mathrm{D}_t \rho + \nabla \cdot \mathbf{J} = \mathrm{D}_t \rho + \mathrm{D}_x J_1 + \mathrm{D}_y J_2 = 0} \quad \text{(on PDE)}$$

  conserved density $\rho$ and flux $\mathbf{J} = (J_1, J_2)$

- Example: Shallow water wave (SWW) equations

$$u_t + u u_x + v u_y - 2\,\Omega v + \frac{1}{2} h \theta_x + \theta h_x = 0$$

$$v_t + u v_x + v v_y + 2\,\Omega u + \frac{1}{2} h \theta_y + \theta h_y = 0$$

$$\theta_t + u \theta_x + v \theta_y = 0$$

$$h_t + h u_x + u h_x + h v_y + v h_y = 0$$

- Typical density-flux pair:

$$\rho^{(5)} = v_x\theta - u_y\theta + 2\Omega\theta$$

$$\mathbf{J}^{(5)} = \frac{1}{2}\begin{pmatrix} 4\Omega u\theta - 2uu_y\theta + 2uv_x\theta - h\theta\theta_y \\ 4\Omega v\theta + 2vv_x\theta - 2vu_y\theta + h\theta\theta_x \end{pmatrix}$$

# Algorithm

- **Step 1: Construct the form of the density**

  The SWW equations are invariant under the scaling symmetries

$$(x, y, t, u, v, \theta, h, \Omega) \rightarrow (\lambda^{-1}x, \lambda^{-1}y, \lambda^{-2}t, \lambda u, \lambda v, \lambda\theta, \lambda h, \lambda^2\Omega)$$

and

$$(x, y, t, u, v, \theta, h, \Omega) \rightarrow (\lambda^{-1}x, \lambda^{-1}y, \lambda^{-2}t, \lambda u, \lambda v, \lambda^2\theta, \lambda^0 h, \lambda^2\Omega)$$

Construct a candidate density, for example,

$$\rho = c_1\Omega\theta + c_2 u_y\theta + c_3 v_y\theta + c_4 u_x\theta + c_5 v_x\theta$$

which is scaling invariant under *both* symmetries.

- **Step 2: Determine the constants $c_i$**

Compute $E = -\mathrm{D}_t \rho$ and remove time derivatives

$$\begin{aligned} E &= -(\frac{\partial \rho}{\partial u_x} u_{tx} + \frac{\partial \rho}{\partial u_y} u_{ty} + \frac{\partial \rho}{\partial v_x} v_{tx} + \frac{\partial \rho}{\partial v_y} v_{ty} + \frac{\partial \rho}{\partial \theta} \theta_t) \\ &= c_4 \theta (u u_x + v u_y - 2\Omega v + \tfrac{1}{2} h \theta_x + \theta h_x)_x \\ &\quad + c_2 \theta (u u_x + v u_y - 2\Omega v + \tfrac{1}{2} h \theta_x + \theta h_x)_y \\ &\quad + c_5 \theta (u v_x + v v_y + 2\Omega u + \tfrac{1}{2} h \theta_y + \theta h_y)_x \\ &\quad + c_3 \theta (u v_x + v v_y + 2\Omega u + \tfrac{1}{2} h \theta_y + \theta h_y)_y \\ &\quad + (c_1 \Omega + c_2 u_y + c_3 v_y + c_4 u_x + c_5 v_x)(u \theta_x + v \theta_y) \end{aligned}$$

Require that

$$\mathcal{L}^{(0,0)}_{u(x,y)} E = \mathcal{L}^{(0,0)}_{v(x,y)} E = \mathcal{L}^{(0,0)}_{\theta(x,y)} E = \mathcal{L}^{(0,0)}_{h(x,y)} E \equiv 0$$

- Solution: $c_1 = 2,\ c_2 = -1,\ c_3 = c_4 = 0,\ c_5 = 1$ gives

$$\boxed{\rho = 2\Omega\theta - u_y\theta + v_x\theta}$$

- **Step 3: Compute the flux $\mathbf{J}$**

$$\begin{aligned} E \;=\; & \theta(u_x v_x + u v_{2x} + v_x v_y + v v_{xy} + 2\Omega u_x \\ & + \tfrac{1}{2}\theta_x h_y - u_x u_y - u u_{xy} - u_y v_y - u_{2y} v \\ & + 2\Omega v_y - \tfrac{1}{2}\theta_y h_x) \\ & + 2\Omega u \theta_x + 2\Omega v \theta_y - u u_y \theta_x \\ & - u_y v \theta_y + u v_x \theta_x + v v_x \theta_y \end{aligned}$$

Apply the 2D homotopy operator:

$$\mathbf{J} = (J_1, J_2) = \mathrm{Div}^{-1} E = (\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} E, \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} E)$$

Compute

$$I_u^{(x)}E = u\frac{\partial E}{\partial u_x} + u_x\frac{\partial E}{\partial u_{2x}} - u\mathrm{D}_x\frac{\partial E}{\partial u_{2x}} + \frac{1}{2}u_y\frac{\partial E}{\partial u_{xy}} - \frac{1}{2}u\mathrm{D}_y\frac{\partial E}{\partial u_{xy}}$$
$$= uv_x\theta + 2\Omega u\theta + \frac{1}{2}u^2\theta_y - uu_y\theta$$

Similarly, compute

$$\begin{aligned} I_v^{(x)}E &= vv_y\theta + \frac{1}{2}v^2\theta_y + uv_x\theta \\ I_\theta^{(x)}E &= \frac{1}{2}\theta^2 h_y + 2\Omega u\theta - uu_y\theta + uv_x\theta \\ I_h^{(x)}E &= -\frac{1}{2}\theta\theta_y h \end{aligned}$$

Next,

$$
\begin{aligned}
J_1 &= \mathcal{H}^{(x)}_{\mathbf{u}(x,y)} E \\
&= \int_0^1 \left( I_u^{(x)} E + I_v^{(x)} E + I_\theta^{(x)} E + I_h^{(x)} E \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \\
&= \int_0^1 \Bigg( 4\lambda\Omega u\theta + \lambda^2 \left( 3uv_x\theta + \frac{1}{2}u^2\theta_y - 2uu_y\theta + vv_y\theta \right. \\
&\qquad \left. + \frac{1}{2}v^2\theta_y + \frac{1}{2}\theta^2 h_y - \frac{1}{2}\theta\theta_y h \right) \Bigg) d\lambda \\
&= 2\Omega u\theta - \frac{2}{3}uu_y\theta + uv_x\theta + \frac{1}{3}vv_y\theta + \frac{1}{6}u^2\theta_y \\
&\quad + \frac{1}{6}v^2\theta_y - \frac{1}{6}h\theta\theta_y + \frac{1}{6}h_y\theta^2
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
J_2 &= \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} E \\
&= 2\Omega v\theta + \frac{2}{3} v v_x \theta - v u_y \theta - \frac{1}{3} u u_x \theta - \frac{1}{6} u^2 \theta_x - \frac{1}{6} v^2 \theta_x \\
&\qquad + \frac{1}{6} h\theta\theta_x - \frac{1}{6} h_x \theta^2
\end{aligned}
$$

Hence,

$$
\mathbf{J} = \frac{1}{6} \begin{pmatrix} 12\Omega u\theta - 4uu_y\theta + 6uv_x\theta + 2vv_y\theta + u^2\theta_y + v^2\theta_y - h\theta\theta_y + h_y\theta^2 \\ 12\Omega v\theta + 4vv_x\theta - 6vu_y\theta - 2uu_x\theta - u^2\theta_x - v^2\theta_x + h\theta\theta_x - h_x\theta^2 \end{pmatrix}
$$

## After removing the curl term (by hand)

$$\tilde{\mathbf{J}}^{(5)} = \frac{1}{2} \begin{pmatrix} 4\Omega u\theta - 2uu_y\theta + 2uv_x\theta - h\theta\theta_y \\ 4\Omega v\theta + 2vv_x\theta - 2vu_y\theta + h\theta\theta_x \end{pmatrix}$$

Needed: Algorithm to remove curl terms!

# Conclusions and Future Work

- Usefulness of the homotopy operator: Integration by parts, $\mathrm{D}_x^{-1}$, and $\mathrm{Div}^{-1}$.
- To do: integration of non-exact expressions.
  Example: $f = u_x v + u v_x + u^2 u_{2x}$
  $\int f\, dx = uv + \int u^2 u_{2x}\, dx.$
- To do: integration of parametrized differential functions.
  Example: $f = a u_x v + b u v_x$
  $\int f\, dx = uv$ if $a = b$.
- To do: various PDEs (other than those of evolution type).
- To do: full implementation in *Mathematica*.

## Software packages in *Mathematica*

Codes are available via the Internet:
URL: http://www.mines.edu/fs_home/whereman/
and via anonymous FTP from mines.edu in directory:
pub/papers/math_cs_dept/software/

## Publications


1. W. Hereman, M. Colagrosso, R. Sayers, A. Ringler, B. Deconinck, M. Nivala, and M. S. Hickman, Continuous and Discrete Homotopy Operators and the Computation of Conservation Laws. In: Differential Equations with Symbolic Computation, Eds.: D. Wang and Z. Zheng, Birkhäuser Verlag, Basel (2005), Chapter 15, pp. 249-285.

2. W. Hereman, Symbolic computation of conservation laws of nonlinear partial differential equations in multi-dimensions, Int. J. Quan. Chem. **106**(1), 278-299 (2006).

3. W. Hereman, B. Deconinck, and L. D. Poole, Continuous and discrete homotopy operators: A theoretical approach made concrete, Math. Comput. Simul. (2006) submitted.