ESTIMATED TRANSVERSALITY IN SYMPLECTIC GEOMETRY AND PROJECTIVE MAPS

DENIS AUROUX

1. Introduction

Since Donaldson's original work [7], approximately holomorphic techniques have proven themselves most useful in symplectic geometry and topology, and various classical constructions from algebraic geometry have been extended to the case of symplectic manifolds [3, 4, 8, 11]. All these results rely on an estimated transversality statement for approximately holomorphic sections of very positive bundles, obtained by Donaldson [7, 8]. However, the arguments require transversality not only for sections but also for their covariant derivatives, which makes it necessary to painstakingly imitate the arguments underlying Thom's classical *strong transversality theorem* for jets.

It is our aim in this paper to formulate and prove a general result of estimated transversality with respect to finite stratifications in jet bundles. The transversality properties obtained in the various above-mentioned papers then follow as direct corollaries of this result, thus allowing some of the arguments to be greatly simplified. The result can be formulated as follows (see §2 and §3 for definitions) :

Theorem 1.1. *Let $(E_k)_{k\gg 0}$ be an asymptotically very ample sequence of locally splittable complex vector bundles over a compact almost-complex manifold (X,J). Let $\mathcal{S}_k$ be asymptotically holomorphic finite Whitney quasi-stratifications of the holomorphic jet bundles $\mathcal{J}^r E_k$. Finally, let $\delta > 0$ be a fixed constant. Then there exist constants K and η such that, given any asymptotically holomorphic sections s_k of E_k over X, there exist asymptotically holomorphic sections σ_k of E_k with the following properties for all $k \geq K$:*

(1) $|\sigma_k - s_k|_{C^{r+1},g_k} < \delta$;

(2) *the jet $j^r\sigma_k$ of σ_k is η-transverse to the quasi-stratification $\mathcal{S}_k$.*

We start by introducing in §2 a general notion of ampleness over an almost-complex manifold. Then, in §3 we define the notion of approximately holomorphic quasi-stratification of a jet bundle. Theorem 1.1 and its one-parameter version are proved in §4. Finally, we discuss applications in §5.

2. Ample bundles over almost-complex manifolds

The most general setup in which one can try to define a notion of ampleness is the following. Let X be a compact $2n$-dimensional manifold (possibly with boundary), endowed with an almost-complex structure J. In order to make estimates, we also endow X with a Riemannian metric g compatible with J (i.e. J is g-antisymmetric).

Definition 2.1. *Given positive constants c and δ, a complex line bundle L over X endowed with a Hermitian metric and a connection ∇^L is* (c,δ)-ample *if its curvature 2-form F_L satisfies the inequalities $iF_L(v,Jv) \geq c\,g(v,v)$ for every tangent vector $v \in TX$, and $\sup|F_L^{0,2}| \leq \delta$.*

A sequence of complex line bundles L_k with metrics and connections is asymptotically very ample *if there exist fixed constants δ and $(C_r)_{r\geq 0}$, and a sequence $c_k \to +\infty$, such that the curvature F_k of L_k satisfies the following properties : (1) $iF_k(v,Jv) \geq c_k\,g(v,v)$ for every tangent vector $v \in TX$; (2) $\sup|F_k^{0,2}| \leq \delta\,c_k^{1/2}$; (3) $\sup|\nabla^r F_k| \leq C_r\,c_k$ $\forall r \geq 0$.*

Most of this definition is a natural extension to the almost-complex setup of the classical notion of ampleness on a complex manifold. Because the notion of holomorphic bundle is not relevant in the case of a non-integrable complex structure, one should allow the curvature to contain a non-trivial $(0,2)$-part. However, because $F_L^{0,2}$ is an obstruction to the existence of holomorphic sections, we need uniform bounds on this quantity in order to hope for the existence of approximately holomorphic sections.

The last condition in the definition seems less natural and should largely be considered as a technical assumption needed to obtain some control over the behavior of sections; it is likely that a suitable argument, possibly involving plurisubharmonic techniques, could allow the bounds to be significantly weakened.

Observe that the curvature of a (c,δ)-ample line bundle over X defines, after multiplication by $\frac{i}{2\pi}$, a J-tame symplectic structure on X with integral cohomology class (J is compatible with this symplectic structure if and only if the curvature is of type $(1,1)$).

Conversely, assume that X carries a J-compatible symplectic form ω with integral cohomology class, and choose g to be the Riemannian metric induced by J and ω. Then there exists a line bundle L with first Chern class $c_1(L) = [\omega]$ and a connection with curvature $-2\pi i\omega$ on L. By construction the line bundle L is $(2\pi, 0)$-ample; moreover, the line bundles $L^{\otimes k}$ with the induced connections are $(2\pi k, 0)$-ample and define an asymptotically very ample sequence of line bundles. This example is by far the most interesting one for applications, but many other situations can be considered as well.

In the rest of this section, we consider an asymptotically very ample sequence of line bundles over X, and study the properties of L_k for large values of k. In order to make the estimates below easier to understand, we rescale the metric by setting $g_k = c_k\,g$, which amounts to dividing by $c_k^{r/2}$ the norm of all r-tensors; the Levi-Civita connection is not affected by this rescaling. The bounds of Definition 2.1 imply that : $iF_k(v,Jv) \geq g_k(v,v)$; $|F_k^{0,2}|_{g_k} = O(c_k^{-1/2})$; $|F_k|_{g_k} = O(1)$; $|\nabla^r F_k|_{g_k} = O(c_k^{-1/2})$ $\forall r \geq 1$. Also observe that $|\nabla^r J|_{g_k} = O(c_k^{-1/2})$ $\forall r \geq 1$. Better bounds on higher-order derivatives are trivially available but we won't need them.

Lemma 2.1. *Let L_k be a sequence of asymptotically very ample line bundles L_k over X, and denote by F_k the curvature of L_k. Let $\omega_k = iF_k$, and let*

c_k be the constants appearing in Definition 2.1. Denote by ∇ the Levi-Civita connection associated to g. Then, for large enough k there exist ω_k-compatible almost-complex structures $\tilde{J}_k$ such that $|\nabla^r(\tilde{J}_k - J)|_{g_k} = O(c_k^{-1/2})$ $\forall r \geq 0$.

Proof. We construct $\tilde{J}_k$ locally; patching together the local constructions in order to obtain a globally defined almost-complex structure still satisfying the same type of bounds is an easy task left to the reader (recall that the space of ω_k-compatible almost-complex structures is pointwise contractible).

Let e_1 be a local tangent vector field of unit g_k-length and with $|\nabla^r e_1|_{g_k} = O(c_k^{-1/2})$ $\forall r \geq 1$ (observe that, because of the rescaling process, (X, g_k) is almost flat for large k). We define $e'_1 = Je_1$, and observe that e'_1 has unit g_k-length (J is g-unitary and hence g_k-unitary) and $\omega_k(e_1, e'_1) \geq 1$. Next, we proceed inductively, assuming that we have defined local vector fields $e_1, e'_1, \dots, e_m, e'_m$ with the following properties for all $i, j \leq m$: $e_1, \dots, e_m$ have unit g_k-length; $\omega_k(e_i, e_j) = \omega_k(e'_i, e'_j) = 0$; $\omega_k(e_i, e'_j) = 0$ if $i \neq j$; $\omega_k(e_i, e'_i) \geq 1$; $e'_i - Je_i \in \mathrm{span}(e_1, e'_1, \dots, e_{i-1}, e'_{i-1})$; $|e'_i - Je_i|_{g_k} = O(c_k^{-1/2})$; $|\nabla^r e_i|_{g_k} = O(c_k^{-1/2})$ and $|\nabla^r e'_i|_{g_k} = O(c_k^{-1/2})$ $\forall r \geq 1$.

We choose e_{m+1} to be a g_k-unit vector field which is ω_k-orthogonal to $e_1, e'_1, \dots, e_m, e'_m$. The bound on $|\nabla^r \omega_k|_{g_k}$ implies that we can choose e_{m+1} in such a way that $|\nabla^r e_{m+1}|_{g_k} = O(c_k^{-1/2})$. Next, we define

$$e'_{m+1} = Je_{m+1} + \sum_{i=1}^{m} \frac{\omega_k(e'_i, Je_{m+1})e_i - \omega_k(e_i, Je_{m+1})e'_i}{\omega_k(e_i, e'_i)}.$$

By construction, $\omega_k(e_i, e'_{m+1}) = \omega_k(e'_i, e'_{m+1}) = 0$ for all $i \leq m$. Moreover, $\omega_k(e_{m+1}, e'_{m+1}) = \omega_k(e_{m+1}, Je_{m+1}) \geq 1$.

Since $\omega_k(Je_i, e_{m+1}) = 0$, and because $\omega_k(Je_i, e_{m+1}) - \omega_k(e_i, Je_{m+1})$ is a component of $\omega_k^{0,2}$, we have $\omega_k(e_i, Je_{m+1}) = O(c_k^{-1/2})$. Similarly, we have $\omega_k(e'_i, Je_{m+1}) = \omega_k(Je_i, Je_{m+1}) + \omega_k(e'_i - Je_i, Je_{m+1})$; the first term differs from $\omega_k(e_i, e_{m+1}) = 0$ by a $(0, 2)$-term and is therefore bounded by $O(c_k^{-1/2})$, while the bound on $e'_i - Je_i$ implies that the second term is also bounded by $O(c_k^{-1/2})$. Therefore we have $\omega_k(e'_i, Je_{m+1}) = O(c_k^{-1/2})$. Finally, using the lower bound on $\omega_k(e_i, e'_i)$ we obtain that $|e'_{m+1} - Je_{m+1}|_{g_k} = O(c_k^{-1/2})$. Finally, it is trivial that $|\nabla^r e'_{m+1}|_{g_k} = O(c_k^{-1/2})$; therefore we can proceed with the induction process.

We now define the almost-complex structure $\tilde{J}_k$ by the identities $\tilde{J}_k(e_i) = e'_i$ and $\tilde{J}_k(e'_i) = -e_i$. By construction, $\tilde{J}_k$ is compatible with ω_k, and the corresponding Riemannian metric $\tilde{g}_k$ admits $e_1, e'_1, \dots, e_n, e'_n$ as an orthonormal frame. The required bounds on $\tilde{J}_k$ immediately follow from the available estimates. ☐

Lemma 2.1 makes it possible to recover the main ingredients of Donaldson theory in the more general setting described here. We now introduce some basic definitions and results, imitating Donaldson's original work and subsequent papers [7, 3].

In what follows, L_k is an asymptotically very ample sequence of line bundles over X, c_k are the same constants as in Definition 2.1, and $g_k = c_k\, g$.

Lemma 2.2. *Near any point $x \in X$, and for any value of k, there exist local complex Darboux coordinates $(z_k^1, \dots, z_k^n) : (X, x) \to (C^n, 0)$ for the symplectic structure $\omega_k = iF_k$, such that, denoting by ψ_k the inverse of the coordinate map, the following bounds hold uniformly in x and k over a ball of fixed g-radius around x : $|z_k^i(y)| = O(dist_{g_k}(x, y))$; $|\nabla^r \psi_k|_{g_k} = O(1)$ $\forall r \geq 1$; and, with respect to the almost-complex structure J on X and the canonical complex structure on $\mathbb{C}^n$, $|\bar{\partial}\psi_k(z)|_{g_k} = O(c_k^{-1/2} + c_k^{-1/2}|z|)$, and $|\nabla^r \bar{\partial}\psi_k(z)|_{g_k} = O(c_k^{-1/2})$ for all $r \geq 1$.*

Proof. The argument is very similar to that used by Donaldson [7], except that one needs to be slightly more careful in showing that the various bounds hold uniformly in k. Fix a point $x \in X$: then we can find a neighborhood U of x and a local coordinate map $\phi : U \to \mathbb{C}^n$, such that U contains a ball of fixed uniform g-radius around x, and such that the expressions of g and J in these local coordinates satisfy uniform bounds independently of x (these uniformity properties follow from the compactness of X). A linear transformation can be used to ensure that the differential of ϕ at the origin is $\mathbb{C}$-linear with respect to J. Next, we rescale the coordinates by $c_k^{1/2}$ to obtain a new coordinate map $\phi_k : U \to \mathbb{C}^n$, in which J coincides with the standard almost-complex structure at the origin and has derivatives bounded by $O(c_k^{-1/2})$, while the expression of g_k is bounded between fixed constants and has derivatives bounded by $O(c_k^{-1/2})$.

Next, we observe that the bound on $|\omega_k|$ and the lower bound on $\omega_k(v, Jv)$ imply that the expression of $\omega_k^{(1,1)}$ at the origin of the coordinate chart is bounded from above and below by uniform constants. Therefore, after composing ϕ_k with a suitable element of $GL(n, \mathbb{C})$, we can assume without affecting the bounds on J and g_k that $(\phi_k^{-1})^*(\omega_k^{(1,1)})$ coincides with the standard Kähler form ω_0 of $\mathbb{C}^n$ at the origin.

Define over $\phi_k(U) \subset \mathbb{C}^n$ the symplectic form $\omega_1 = (\phi_k^{-1})^*\omega_k$. By construction, $\omega_1(0) - \omega_0(0) = O(c_k^{-1/2})$. Observe that, in the chosen coordinates, the Levi-Civita connection of g_k differs from the trivial connection by $O(c_k^{-1/2})$; therefore, the bounds on $|\nabla^r \omega_k|_{g_k}$ imply that the derivatives of ω_1 are also bounded by $O(c_k^{-1/2})$, and that $|\omega_1(z) - \omega_0(z)| = O(c_k^{-1/2} + c_k^{-1/2}|z|)$.

In particular, decreasing the size of U by at most a fixed factor if necessary, we obtain that the closed 2-forms $\omega_t = t\omega_1 + (1-t)\omega_0$ over $\phi_k(U)$ are all symplectic, and we can apply Moser's argument to construct in a controlled way a symplectomorphism between a subset of $(\phi_k(U), \omega_1)$ and a subset of $(\mathbb{C}^n, \omega_0)$. More precisely, it follows immediately from Poincaré's lemma that we can choose a 1-form α such that $\omega_1 - \omega_0 = d\alpha$, and such that $\alpha(0) = 0$, $|\alpha(z)| = O(c_k^{-1/2}|z| + c_k^{-1/2}|z|^2)$, $|\nabla\alpha(z)| = O(c_k^{-1/2} + c_k^{-1/2}|z|)$ and $|\nabla^r \alpha(z)| = O(c_k^{-1/2})$ $\forall r \geq 2$. Next, we define vector fields X_t by the identity $i_{X_t}\omega_t = \alpha$; clearly X_t and its derivatives satisfy the same bounds as α.

Integrating the flow of the vector fields X_t we obtain diffeomorphisms ρ_t, and it is a classical fact that the map $\tilde{\phi}_k = \rho_1 \circ \phi_k$ is a local symplectomorphism between (X, ω_k) and $(\mathbb{C}^n, \omega_0)$ and therefore defines Darboux coordinates. Because $|z| = O(c_k^{1/2})$ over a ball of fixed g-radius around x, the vector fields X_t satisfy a uniform bound of the type $|X_t(z)| \le \lambda|z|$ for some constant λ, so that $|\rho_t(z)| \le e^{\lambda t}|z|$, and therefore $\tilde{\phi}_k$ is well-defined over a ball of fixed g-radius around x. Moreover, the bounds $|\nabla(\rho_1 - \mathrm{Id})| = O(c_k^{-1/2} + c_k^{-1/2}|z|)$, obtained by integrating the bounds on $\nabla\alpha$, and $|\bar\partial(\phi_k^{-1})| = O(c_k^{-1/2}|z|)$, obtained from the bounds on the expression on J in the local coordinates, imply that $|\bar\partial(\tilde{\phi}_k^{-1})| = O(c_k^{-1/2} + c_k^{-1/2}|z|)$. Similarly, the bounds $|\nabla^{r+1}\rho_1| = O(c_k^{-1/2})$ and $|\nabla^r\bar\partial(\phi_k^{-1})| = O(c_k^{-1/2})$ for all $r \ge 1$ imply that $|\nabla^r\bar\partial(\tilde{\phi}_k^{-1})| = O(c_k^{-1/2})$. This completes the proof of Lemma 2.2. ☐

Definition 2.2. *A family of sections of L_k is* asymptotically J-holomorphic *for $k \to \infty$ if there exist constants $(C_r)_{r\ge 0}$ such that every section $s \in \Gamma(L_k)$ in the family satisfies at every point of X the bounds $|\nabla^r s|_{g_k} \le C_r$ and $|\nabla^r \bar\partial_J s|_{g_k} \le C_r\, c_k^{-1/2}$ for all $r \ge 0$, where $\bar\partial_J$ is the $(0,1)$-part of the connection on L_k.*

A family of sections of L_k has uniform Gaussian decay *properties if there exist a constant $\lambda > 0$ and polynomials $(P_r)_{r\ge 0}$ with the following property : for every section s of L_k in the family, there exists a point $x \in X$ such that for all $y \in X$ and for all $r \ge 0$, $|\nabla^r s(y)|_{g_k} \le P_r(d_k(x,y)) \exp(-\lambda\, d_k(x,y)^2)$, where $d(.,.)$ is the distance induced by g_k.*

Lemma 2.3. *For all large enough values of k and for every point $x \in X$, there exists a section $s^{\mathrm{ref}}_{k,x}$ of L_k with the following properties : (1) the family of sections $(s^{\mathrm{ref}}_{k,x})_{x\in X, k\gg 0}$ is asymptotically J-holomorphic; (2) the family $(s^{\mathrm{ref}}_{k,x})_{x\in X, k\gg 0}$ has uniform Gaussian decay properties, each section $s^{\mathrm{ref}}_{k,x}$ being concentrated near the point x; (3) there exists a constant $\kappa > 0$ independent of x and k such that $|s^{\mathrm{ref}}_{k,x}| \ge \kappa$ at every point of the ball of g_k-radius 1 centered at x.*

Proof. The argument is a direct adaptation of the proof of Proposition 11 in Donaldson's paper [7]. Pick a value of k and a point $x \in X$. We work in the approximately J-holomorphic Darboux coordinates given by Lemma 2.2, and use a trivialization of L_k in which the connection 1-form becomes $\frac{1}{4}\sum(z_j d\bar z_j - \bar z_j dz_j)$. Then, we define a local section of L_k by $s(z) = \exp(-\frac{1}{4}|z|^2)$ and observe that s is holomorphic with respect to the standard complex structure of $\mathbb{C}^n$. Multiplying s by a cut-off function which equals 1 over the ball of radius $c_k^{1/6}$ around the origin, we obtain a globally defined section of L_k; because of the estimates on the Darboux coordinates one easily checks that the families of sections constructed in this way are asymptotically holomorphic and have uniform Gaussian decay properties [7]. ☐

We are also interested in working with higher rank bundles. The definition of ampleness becomes the following :

Definition 2.3. *A sequence of complex vector bundles E_k with metrics and connections is* asymptotically very ample *if there exist constants δ, $(C_r)_{r\ge 0}$, and $c_k \to +\infty$, such that the curvature F_k of E_k satisfies the following properties :* (1) $\langle iF_k(v,Jv).u,u\rangle \ge c_k\, g(v,v)\,|u|^2$, $\forall v\in TX$, $\forall u \in E_k$; (2) $\sup|F_k^{0,2}|_g \le \delta_r\, c_k^{1/2}$; (3) $\sup|\nabla^r F_k|_g \le C_r\, c_k$ $\forall r\ge 0$.

A sequence of asymptotically very ample complex vector bundles E_k with metrics $|.|_k$ and connections ∇_k is locally splittable *if, given any point $x\in X$, there exists over a ball of fixed g-radius around x a decomposition of E_k as a direct sum $L_{k,1}\oplus\cdots\oplus L_{k,m}$ of line bundles, such that the following properties hold :* (1) *the $|.|_k$-determinant of a local frame consisting of unit length local sections of $L_{k,1},\dots,L_{k,m}$ is bounded from below by a fixed constant independently of x and k;* (2) *denoting by $\nabla_{k,i}$ the connection on $L_{k,i}$ obtained by projecting $\nabla_{k|L_{k,i}}$ to $L_{k,i}$, and by ∇'_k the direct sum of the $\nabla_{k,i}$, the 1-form $\alpha_k=\nabla_k-\nabla'_k$ (the non-diagonal part of ∇_k) satisfies the uniform bounds $|\nabla^r\alpha_k|_g = O(c_k^{r/2})$ $\forall r\ge 0$ independently of x.*

For example, if E is a fixed complex vector bundle and L_k are asymptotically very ample line bundles, then the vector bundles $E\otimes L_k$ are locally splittable and asymptotically very ample; so are direct sums of vector bundles of this type.

Observe that, if E_k is an asymptotically very ample sequence of locally splittable vector bundles, then near any given point $x\in X$ the summands $L_{k,1},\dots,L_{k,m}$ are asymptotically very ample line bundles. Therefore, by Lemma 2.3 they admit asymptotically holomorphic sections $s^{\mathrm{ref}}_{k,x,i}$ with uniform Gaussian decay away from x. Moreover, these sections, which define a local frame for E_k, are easily checked to be asymptotically J-holomorphic not only as sections of $L_{k,i}$ but also as sections of E_k.

3. Estimated transversality in jet bundles

3.1. **Asymptotically holomorphic stratifications.**

Throughout this section, we will denote by F_k be a sequence of complex vector bundles over X, or more generally fiber bundles with almost-complex manifolds as fibers. We also fix, in a manner compatible with the almost-complex structures J^v of the fibers, metrics g^v on the fibers of F_k and a connection on F_k. Finally, we fix a sequence of constants $c_k\to+\infty$.

The connection on F_k induces a splitting $TF_k = T^vF_k\oplus T^hF_k$ between horizontal and vertical tangent spaces ; this splitting makes it possible to define a metric $\hat g_k$ and an almost-complex structure $\hat J_k$ on the total space of F_k, obtained by orthogonal direct sum of g^v and J^v on T^vF_k together with the pullbacks of J and $g_k = c_k\, g$ on $T^hF_k\simeq \pi^*TX$.

We want to consider approximately holomorphic stratifications of the fibers of F_k, depending in an approximately holomorphic way on the point in the base manifold X. For simplicity, we assume that the topological picture is the same in every fiber of F_k, i.e. we restrict ourselves to stratifications which are everywhere transverse to the fibers. We will denote the strata by $(S_k^a)_{a\in A_k}$; we assume that the number of strata is finite. Each S_k^a is a possibly non-closed submanifold in F_k, whose closure is obtained by adding other lower dimensional strata : writing $b\prec a$ iff S_k^b is contained in $\overline{S_k^a}$, we have

$$\partial S_k^a \stackrel{\text{def}}{=} \overline{S_k^a} - S_k^a = \bigcup_{b \prec a} S_k^b.$$

We only consider *Whitney stratifications* ; in particular, transversality to a given stratum S_k^a implies transversality over a neighborhood of S_k^a to all the strata whose closure contains S_k^a, i.e. all the S_k^b for $b \succ a$. Also note that we discard any open strata, as they are irrelevant for transversality purposes ; so each S_k^a has codimension at least 1.

Definition 3.1. *Let (M, J) be an almost-complex manifold, with a Riemannian metric, and let s be a complex-valued function over M or a section of an almost-complex bundle with metrics and connection. Given two constants C and c, we say that s is* C^2-approximately holomorphic with bounds (C, c), *or* C^2-AH(C, c), *if it satisfies the following estimates :*

$$|s| + |\nabla s| + |\nabla\nabla s| \le C, \qquad |\bar\partial s| + |\nabla\bar\partial s| \le C\, c^{-1/2}.$$

Moreover, given constants $c_k \to +\infty$, we say that a sequence $(s_k)_{k\gg 0}$ of functions or sections is C^2-asymptotically holomorphic, *or* C^2-AH, *if there exists a fixed constant C such that each section s_k is C^2-$AH(C, c_k)$.*

Definition 3.2. *Let F_k be a sequence of almost-complex bundles over X, endowed with metrics and connections as above. For all values of k, let $(S_k^a)_{a\in A_k}$ be finite Whitney stratifications of F_k ; assume that the total number of strata is bounded by a fixed constant independently of k, and that all strata are transverse to the fibers of F_k.*

We say that this sequence of stratifications is asymptotically holomorphic *if, given any bounded subset $U_k \subset F_k$, and for every $\epsilon > 0$, there exist positive constants C_ϵ and ρ_ϵ depending only on ϵ and on the size of the subset U_k but not on k, with the following property. For every point $x \in U_k$ lying in a certain stratum S_k^a and at $\hat g_k$-distance greater than ϵ from $\partial S_k^a = \overline{S_k^a} - S_k^a$, there exist complex-valued functions $f_1, \dots, f_p$ over the ball $B = B_{\hat g_k}(x, \rho_\epsilon)$ with the following properties :*

(1) *a local equation of S_k^a over B is $f_1 = \dots = f_p = 0$;*

(2) *$|df_1 \wedge \dots \wedge df_p|_{\hat g_k}$ is bounded from below by ρ_ϵ at every point of B ;*

(3) *the restrictions of f_i to each fiber of F_k near x are C^2-$AH(C_\epsilon, c_k)$;*

(4) *for any constant $\lambda > 0$, and for any local section s of F_k which is C^2-$AH(\lambda, c_k)$ with respect to the metric g_k on X and which intersects non-trivially the ball B, the function $f_i \circ s$ is C^2-$AH(\lambda C_\epsilon, c_k)$; moreover, given a local C^2-$AH(\lambda, c_k)$ section θ of $s^*T^vF_k$, the functions $df_i \circ \theta$ are C^2-$AH(\lambda C_\epsilon, c_k)$;*

(5) *at every point $y \in B$ belonging to a stratum S_k^b such that $S_k^a \subset \partial S_k^b$, the norm of the orthogonal projection onto the normal space $N_y S_k^b$ of any unit length vector $v \in T_y F_k$ such that $df_1(v) = \dots = df_p(v) = 0$ is bounded by $C_\epsilon \operatorname{dist}_{\hat g_k}(y, S_k^a)$.*

These conditions on the stratification can be reformulated more geometrically as follows. First, the strata must be uniformly transverse to the fibers of F_k, i.e. one requires the minimum angle [11] between TS_k^a and T^vF_k to be bounded from below. Second, the submanifolds $S_k^a \subset F_k$ must be asymptotically $\hat J_k$-holomorphic, i.e. $\hat J_k(TS_k^a)$ and TS_k^a lie within $O(c_k^{-1/2})$ of each other. Third, the curvature of S_k^a as a submanifold of F_k must be uniformly bounded. Finally, the quantity measuring the lack of $\hat{J}_k$-holomorphicity of S_k^a must similarly vary in a controlled way.

We finish this section by introducing the notion of estimated transversality between a section and a stratification. Observe that, given any submanifold $N \subset M$, we can define over a neighborhood of N a "parallel" distribution $D_N \subset TM$ by parallel transport of TN in the normal direction to N. Also recall that the *minimum angle* between two linear subspaces U and V is defined as the minimum angle between a vector orthogonal to U and a vector orthogonal to V [11]. The minimum angle between U and V is non-zero if and only if they are transverse to each other, and in that case it can also be defined as the minimum angle between non-zero vectors orthogonal to $U \cap V$ in U and V.

Definition 3.3. *Given a constant $\eta > 0$, we say that a section s of a vector bundle carrying a metric and a connection is η-transverse to* 0 *if, at every point x such that $|s(x)| \leq \eta$, the covariant derivative $\nabla s(x)$ is surjective and admits a right inverse of norm less than η^{-1}.*

Fix a constant $\eta > 0$, and a section s of a bundle carrying a metric and a finite Whitney stratification $\mathcal{S} = (S^a)_{a\in A}$ everywhere transverse to the fibers. We say that s is η-transverse to the stratification $\mathcal{S}$ if, at every point where s lies at distance less than η from some stratum S^a, the graph of s is transverse to the parallel distribution D_{S^a}, with a minimum angle greater than η.

Finally, we say that a sequence of sections is uniformly tranverse to 0 *(resp. to a sequence of stratifications) if there exists a fixed constant $\eta > 0$ such that all sections in the sequence are η-transverse to* 0 *(resp. the stratifications).*

Note that the above condition of transversality of the section s to each stratum S^a is only well-defined outside of a small neighborhood of the lower-dimensional strata contained in ∂S^a ; however, near these strata the assumption that $\mathcal{S}$ is Whitney makes transversality to S^a a direct consequence of the η-transversality to the lower-dimensional strata.

Another way in which uniform transversality to a stratification can be formulated is to use local equations of the strata, as in Definition 3.2. One can then define η-transversality as follows : at every point where s lies at distance less than η from S^a, and considering local equations $f_1 = \cdots = f_p = 0$ of S^a such that each $|df_i|$ is bounded by a fixed constant and $|df_1 \wedge \cdots \wedge df_p|$ is bounded from below by a fixed constant, the function $(f_1 \circ s, \dots, f_p \circ s)$ with values in $\mathbb{C}^p$ must be η-transverse to 0. The two definitions are equivalent up to changing the constant η by at most a bounded factor.

3.2. **Quasi-stratifications in jet bundles.** Let E_k be an asymptotically very ample sequence of locally splittable rank m vector bundles over the compact almost-complex manifold (X, J). We can introduce the *holomorphic jet bundles*

$$\mathcal{J}^r E_k = \bigoplus_{j=0}^{r} (T^*X^{(1,0)})^{\otimes j}_{\mathrm{sym}} \otimes E_k.$$

More precisely, the holomorphic part of the r-jet of a section s of E_k is defined inductively as follows : $T^*X^{(1,0)}$ and E_k, as complex vector bundles carrying a connection over an almost-complex manifold, are endowed with ∂ operators (the $(1,0)$ part of the connection) ; the r-jet of s is $j^r s = (s, \partial_{E_k} s, \partial_{T^*X^{(1,0)} \otimes E_k} (\partial_{E_k} s)_{\rm sym}, \dots)$.

Observe that, because the almost-complex structure J is not integrable and because the curvature of E_k is not necessarily of type $(1,1)$, the derivatives of order ≥ 2 are not symmetric tensors, but rather satisfy equality relations involving curvature terms and lower-order derivatives. However, we will only consider the symmetric part of the jet ; for example, the 2-tensor component of $j^r s$ is defined by $(\partial\partial s)_{\rm sym}(u,v) = \frac{1}{2}(\langle \partial(\partial s), u \otimes v\rangle + \langle \partial(\partial s), v \otimes u\rangle)$. Note that, anyway, in the case of asymptotically holomorphic sections, the antisymmetric terms are bounded by $O(c_k^{-1/2})$, because the $(2,0)$ curvature terms and Nijenhuis tensor are bounded by $O(c_k^{-1/2})$.

The metrics and connections on TX and on E_k naturally induce Hermitian metrics and connections on $\mathcal{J}^r E_k$ (to define the metric we use the rescaled metric g_k on X). In fact, it is easy to see that the vector bundles $\mathcal{J}^r E_k$ are asymptotically very ample.

Recall that, near any given point $x \in X$, there exist local approximately holomorphic coordinates ; besides a local identification of X with $\mathbb{C}^n$, these coordinates also provide an identification of $T^*X^{(1,0)}$ with $T^*\mathbb{C}^{n\,(1,0)}$. Moreover, by Lemma 2.3 there exist asymptotically holomorphic sections $s^{\rm ref}_{k,x,i}$ of E_k with Gaussian decay away from x and defining a local frame in E_k. Using these sections to trivialize E_k, we can locally identify $\mathcal{J}^r E_k$ with a space of jets of holomorphic $\mathbb{C}^m$-valued maps over $\mathbb{C}^n$. Observe however that, when we consider the holomorphic parts of jets of approximately holomorphic sections of E_k, the integrability conditions normally satisfied by jets of holomorphic functions only hold in an approximate sense.

In general, the various possible choices of trivializations of $\mathcal{J}^r E_k$ differ by approximately holomorphic diffeomorphisms of $\mathbb{C}^n$ and also by the action of approximately holomorphic local sections of the automorphism bundle $\mathrm{GL}(E_k)$. However, when E_k is of the form $\mathbb{C}^m \otimes L_k$ where L_k is a line bundle, the only automorphisms of E_k which we need to consider are multiplications by complex-valued functions.

Denote by $\mathcal{J}^r_{n,m}$ the space of r-jets of holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^m$: pointwise, the identifications of the fibers of $\mathcal{J}^r E_k$ with $\mathcal{J}^r_{n,m}$ given by local trivializations differ from each other by the action of $GL_n(\mathbb{C}) \times GL_m(\mathbb{C})$ (or $GL_n(\mathbb{C}) \times \mathbb{C}^*$ when $E_k = \mathbb{C}^m \otimes L_k$), where $GL_n(\mathbb{C})$ corresponds to changes in the identification of $T^*X^{(1,0)}$ with $T^*\mathbb{C}^{n\,(1,0)}$ and $GL_m(\mathbb{C})$ or $\mathbb{C}^*$ corresponds to changes in the trivialization of E_k. Some stratifications of $\mathcal{J}^r_{n,m}$ are invariant under the actions of $GL_n(\mathbb{C})$ and $GL_m(\mathbb{C})$ (resp. $\mathbb{C}^*$). Given such a stratification it becomes easy to construct an asymptotically holomorphic sequence of finite Whitney stratifications of $\mathcal{J}^r E_k$, modelled in each fiber on the given stratification of $\mathcal{J}^r_{n,m}$. Many important examples of asymptotically holomorphic stratifications, and in a certain sense all the geometrically relevant ones, are obtained by this construction (see Proposition 3.1 below).

We also wish to consider cases where the available structure is not exactly a Whitney stratification but behaves in a similar manner with respect to transversality. We call such a structure a "Whitney quasi-stratification".

Given a submanifold $S \subset \mathcal{J}^r_{n,m}$, one can introduce the subset Θ_S of all points $\sigma \in S$ such that there exists a holomorphic $(r+1)$-jet whose r-jet component is σ and which, considered as a 1-jet of r-jets, intersects S transversely at σ. For example, if S is the subset of all jets $(\sigma_0, \dots, \sigma_r)$ such that $\sigma_0 = 0$, the subset Θ_S consists of those jets such that $\sigma_0 = 0$ and σ_1 is surjective.

Similarly, when S is a submanifold in $\mathcal{J}^r E_k$, we can view an element of $\mathcal{J}^{r+1} E_k$ as the holomorphic 1-jet of a section of $\mathcal{J}^r E_k$. More precisely, for any point $x \in X$, we can associate to any $\sigma = (\sigma_0, \dots, \sigma_{r+1}) \in (\mathcal{J}^{r+1} E_k)_x$ the 1-jet at x of a local section $\tilde{\sigma}$ of $\mathcal{J}^r E_k$, such that $\tilde{\sigma}(x) = (\sigma_0, \dots, \sigma_r)$, $(\partial\tilde{\sigma}(x))^{\mathrm{sym}} = (\sigma_1, \dots, \sigma_{r+1})$, $(\partial\tilde{\sigma}(x))^{\mathrm{antisym}} = 0$, and $\bar{\partial}\tilde{\sigma}(x) = 0$ (in this definition, $\partial\tilde{\sigma}(x) \in T^*X^{1,0} \otimes (\bigoplus (T^*X^{1,0})^{\otimes j}_{\mathrm{sym}} \otimes E)$ is decomposed into a symmetric part and an antisymmetric part). Then, we define Θ_S as the set of points of S for which there exists an element $\sigma \in \mathcal{J}^{r+1} E_k$ such that the corresponding 1-jet $\tilde{\sigma}$ in $\mathcal{J}^r E_k$ intersects S transversely at the given point. For example, if S is the set of r-jets $(\sigma_0, \dots, \sigma_r)$ such that $\sigma_0 = 0$, then Θ_S is the set of r-jets such that $\sigma_0 = 0$ and σ_1 is surjective. Also observe that Θ_S is always empty when the codimension of S is greater than n.

Definition 3.4. *Given a finite set $(A, \prec)$ carrying a binary relation without cycles (i.e., $a_1 \prec \dots \prec a_p \Rightarrow a_p \not\prec a_1$), a finite Whitney quasi-stratification of $\mathcal{J}^r_{n,m}$ indexed by A is a collection $(S^a)_{a \in A}$ of smooth submanifolds of $\mathcal{J}^r_{m,n}$, not necessarily mutually disjoint, with the following properties :* (1) *$\partial S^a = \overline{S^a} - S^a \subseteq \bigcup_{b \prec a} S^b$;* (2) *given any point $p \in \partial S^a$, there exists $b \prec a$ such that $p \in S^b$ and such that either $p \notin \Theta_{S^b}$ or $S^b \subset \partial S^a$ and the Whitney regularity condition is satisfied at all points of S^b.*

Similarly, we can define the notion of asymptotically holomorphic finite Whitney quasi-stratifications of $\mathcal{J}^r E_k$. This is similar to Definition 3.2, except that the collections (S^a_k) are quasi-stratifications rather than stratifications, i.e. $\partial S^a_k \subseteq \bigcup_{b \prec a} S^b_k$, and for every $p \in \partial S^a_k$ there exists $b \prec a$ such that either $p \in S^b_k - \Theta_{S^b_k}$ or $p \in S^b_k \subset \partial S^a_k$; in the latter case the Whitney condition is required. Also, observe that condition (5) in Definition 3.2 is only required in the second case, and not for all b such that $a \prec b$.

It is important to understand that the notion of quasi-stratification is merely an attempt at simplifying the framework for applications of Theorem 1.1. In fact, most quasi-stratifications can be refined into genuine stratifications by suitably subdividing the strata into smaller pieces. However, by definition these modifications occur at points of $\mathcal{J}^r E_k$ that no generic jet can hit, thus making them utterly irrelevant to transversality.

Proposition 3.1. *Let $\mathcal{S} = (S^a)_{a \in A}$ be a finite Whitney quasi-stratification of $\mathcal{J}^r_{n,m}$ by complex submanifolds, invariant under the action of $GL_n(\mathbb{C}) \times GL_m(\mathbb{C})$ or $GL_n(\mathbb{C}) \times \mathbb{C}^*$. Let E_k be an asymptotically very ample sequence of rank m complex vector bundles over X, trivialized near every point by suitable choices of local asymptotically holomorphic coordinates and sections. Assume that $\mathcal{S}_k = (S^a_k)_{a \in A}$ are quasi-stratifications of $\mathcal{J}^r E_k$ such that, in each local trivialization, the intersection of S^a_k with every fiber becomes identified with S^a. Then the sequence of quasi-stratifications $\mathcal{S}_k$ is asymptotically holomorphic.*

The proof of this result is easy and left to the reader ; the independence on k of the model holomorphic quasi-stratification of $\mathcal{J}^r_{n,m}$ and the availability of asymptotically holomorphic local trivializations of $\mathcal{J}^r E_k$ (Lemma 2.2 and Lemma 2.3) immediately yield the necessary estimates on the strata of $\mathcal{S}_k$. The only important point to observe is that, because the strata of $\mathcal{S}$ are $GL_m(\mathbb{C})$-invariant (resp. $\mathbb{C}^*$-invariant), the local trivializations identifying S^a_k with S^a also identify $\Theta_{S^a_k}$ with Θ_{S^a}. This is e.g. due to the fact that, up to a suitable change in the choice of the local coordinates on X and local reference sections of L_k, i.e. up to a local gauge transformation, we can assume that the connection on $\mathcal{J}^r E_k$ agrees at a given point $x \in X$ with the trivial connection on $\mathcal{J}^r_{n,m}$; above x, the identification of $(r+1)$-jets with 1-jets of r-jets then becomes the same in $\mathcal{J}^r E_k$ as in $\mathcal{J}^r_{n,m}$, so that the definitions of Θ_S in $\mathcal{J}^r_{n,m}$ and in $\mathcal{J}^r E_k$ agree with each other.

Various examples of applications of Proposition 3.1 will be given in §5.

Finally, we state a one-parameter version of Theorem 1.1. Consider a continuous one-parameter family $(J_t)_{t\in[0,1]}$ of almost-complex structures on X, and a one-parameter family of asymptotically holomorphic finite Whitney (quasi)-stratifications $(\mathcal{S}_{k,t})_{k\gg 0, t\in[0,1]}$ of almost-complex bundles $F_{k,t}$ over (X, J_t). We say that the (quasi)-stratifications $\mathcal{S}_{k,t}$ depend continuously on t if the following one-parameter version of Definition 3.2 is true : for every $\epsilon > 0$, there exist constants ρ_ϵ and C_ϵ with the following property. Given any continuous path $(x_t)_{t\in[t_1,t_2]}$ of points all belonging to the fibers of $F_{k,t}$ above a same point in X, and assuming that all the points x_t belong to certain strata $S^a_{k,t}$ while lying at distance more than ϵ from $\partial S^a_{k,t}$, there exist for all $t \in [t_1, t_2]$ complex-valued functions $f_{1,t}, \dots, f_{p,t}$ defined over the ball $B_{\hat{g}_{k,t}}(x_t, \rho_\epsilon)$ and depending continuously on t, satisfying the various properties of Definition 3.2 for all values of t.

With this understood, the result is the following :

Theorem 3.2. *Let $(J_t)_{t\in[0,1]}$ be a continuous one-parameter family of almost-complex structures on the compact manifold X, and let $(E_{k,t})_{k\gg 0, t\in[0,1]}$ be a family of complex vector bundles over X endowed with metrics and connections depending continuously on t and such that the sequence $E_{k,t}$ is asymptotically very ample and locally splittable over (X, J_t) for all t. Let $\mathcal{S}_{k,t}$ be asymptotically holomorphic finite Whitney quasi-stratifications of $\mathcal{J}^r E_{k,t}$ depending continuously on t. Finally, let $\delta > 0$ be a fixed constant. Then there exist constants K and η such that, given any one-parameter family of asymptotically holomorphic sections $s_{k,t}$ of $E_{k,t}$ over X depending continuously on t, there exist asymptotically holomorphic sections $\sigma_{k,t}$ of $E_{k,t}$, depending continuously on t, with the following properties for all $k \geq K$ and for all $t \in [0,1]$:*

(1) $|\sigma_{k,t} - s_{k,t}|_{C^{r+1}, g_k} < \delta$;

(2) *the jet $j^r\sigma_{k,t}$ of $\sigma_{k,t}$ is η-transverse to $\mathcal{S}_{k,t}$.*

4. Proof of the main result

The proof of Theorem 1.1 is quite similar to the arguments in previous papers [3, 4, 7, 8, 11]. It relies heavily on the fact that the estimated transversality of the r-jet of a section to a given submanifold of the jet bundle is a local and C^{r+1}-open property in the following sense [3]. Given a submanifold S of $\mathcal{J}^r E_k$, a constant $\eta > 0$ and a point $x \in X$, say that a section s of E_k satisfies the property $\mathcal{P}(S, \eta, x)$ if either the r-jet $j^r s(x)$ lies at distance more than η from S or $j^r s$ is η-transverse to S at x (in the sense of Definition 3.3, i.e. the minimum angle between the graph of $j^r s$ and the parallel distribution to S is at least η). The property $\mathcal{P}(S, \eta, x)$ depends only on the $(r+1)$-jet of s at x ("locality"). Moreover, if s satisfies $\mathcal{P}(S, \eta, x)$, then any section σ such that $|j^{r+1}\sigma(x) - j^{r+1}s(x)| < \epsilon$ satisfies $\mathcal{P}(S, \eta - C\epsilon, x)$, where C is some fixed constant involving only the curvature bounds of S ("openness").

A first consequence is that Theorem 1.1 can be proved by successively perturbing the given sections s_k in order to ensure transversality to the various strata. To show this, we first remark that, given any index b, the uniform transversality of $j^r s_k$ to all the strata S_k^a with $a \prec b$ implies its uniform transversality to S_k^b over a neighborhood of ∂S_k^b.

Indeed, first consider a pair of indices $a \prec b$ such that $S_k^a \subset \partial S_k^b$. By condition (5) of Definition 3.2, near a point of S_k^a the tangent space to S_k^b almost contains the parallel distribution to TS_k^a ; therefore, there exists a constant κ (independent of a and b) such that, for any small $\alpha > 0$, the α-transversality of $j^r s_k$ to S_k^a implies its $\frac{\alpha}{4}$-transversality to S_k^b over the $\kappa\alpha$-neighborhood of S_k^a. Next, consider a pair of indices $a \prec b$ and a point $p \in \partial S_k^b \cap (S_k^a - \Theta_{S_k^a})$: in this case, if the graph of $j^r s_k$ is α-transverse to S_k^a but intersects the ball of radius $\frac{\alpha}{2}$ around p, we can find an approximately holomorphic section σ_k of E_k differing from s_k by less than $\frac{3\alpha}{4}$ and whose jet goes through p. By definition of $\Theta_{S_k^a}$, all lifts of p in $\mathcal{J}^{r+1}E_k$, including $j^{r+1}\sigma_k$, correspond to local sections which intersect S_k^a non-transversely; because the antisymmetric and antiholomorphic terms in $\nabla(j^r\sigma_k)$ are smaller than $O(c_k^{-1/2})$, the minimum angle between $j^r\sigma_k$ and S_k^a at p is bounded by $O(c_k^{-1/2})$. However, since σ_k is close to s_k, its r-jet should be $\frac{\alpha}{4}$-transverse to S_k^a, which gives a contradiction. Therefore, $j^r s_k$ remains at distance more than $\frac{\alpha}{2}$ from p; this implies the $\frac{\alpha}{4}$-transversality to S_k^b of $j^r s_k$ over the $\frac{\alpha}{4}$-neighborhood of every point of $(S_k^a - \Theta_{S_k^a}) \cap \partial S_k^b$. Since these are the only two possible cases near the boundary of S_k^b, the uniform transversality of $j^r s_k$ to S_k^a for all $a \prec b$ implies its uniform transversality to S_k^b near ∂S_k^b.

Now, extend the binary relation $\prec$ on the set of strata of each $\mathcal{S}_k$ into a total order relation $<$, so that the indices in A_k can be identified with integers and the closure of a given stratum consists only of strata appearing before it. Assume that a first perturbation by less than $\delta_0 = \frac{\delta}{2}$ makes it possible to obtain for large k the η_1-transversality of $j^r s_k$ to the first stratum S_k^1, for some constant η_1 independent of k. Next, let δ_1 be a constant sufficiently smaller than δ and η_1 (but independent of k), and assume that a perturbation by at most δ_1 allows us to obtain the η_2-transversality of $j^r s_k$ to the second stratum S_k^2 outside of the $\frac{1}{4}\kappa\eta_1$-neighborhood of ∂S_k^2, for some constant η_2. Because this new perturbation is small enough, the resulting sections remain $\frac{\eta_1}{2}$-transverse to S_k^1 ; also, by the above observation this automatically implies the estimated transversality to S_k^2 of $j^r s_k$ near the points of $\partial S_k^2 \subseteq S_k^1$.

We can continue in this way until all strata have been considered ; each perturbation added to ensure estimated transversality to a new stratum outside of a small fixed size neighborhood of its boundary is chosen small enough in order not to affect the previously obtained transversality properties.

The fact that estimated transversality is local and open also makes it possible to reduce to a purely local setup, using a globalization principle due to Donaldson [7] and which can be formulated as follows (Proposition 3 of [3]) :

Proposition 4.1. *Let $\mathcal{P}_k(\eta, x)_{x\in X,\eta>0,k\gg 0}$ be local and C^{r+1}-open properties of sections of E_k over X. Assume that there exist constants c, c' and ν such that, given any $x \in X$, any small enough $\delta > 0$, and asymptotically holomorphic sections s_k of E_k, there exist, for all large enough k, asymptotically holomorphic sections $\tau_{k,x}$ of E_k with the following properties :* (a) $|\tau_{k,x}|_{C^{r+1},g_k} < \delta$, (b) *the sections $\frac{1}{\delta}\tau_{k,x}$ have uniform Gaussian decay away from x, and* (c) *the sections $s_k + \tau_{k,x}$ satisfy the property $\mathcal{P}_k(\eta, y)$ for all $y \in B_{g_k}(x, c)$, with $\eta = c'\delta\log(\delta^{-1})^{-\nu}$.*

Then, given any $\alpha > 0$ and asymptotically holomorphic sections s_k of E_k, there exist, for all large enough k, asymptotically holomorphic sections σ_k of E_k such that $|s_k - \sigma_k|_{C^{r+1},g_k} < \alpha$ and the sections σ_k satisfy $\mathcal{P}_k(\epsilon, x)$ $\forall x \in X$ for some $\epsilon > 0$ independent of k.

Proposition 4.1 is in fact slightly stronger than the previous results, as the notion of asymptotic holomorphicity has been extended to a more general framework in §2, but the argument remains strictly the same.

With this result, we are reduced to the problem of finding a localized perturbation of s_k near a given point x in order to ensure transversality to a given stratum. More precisely, fix an index $a \in A_k$ in each stratification, and remember that, from the previous steps of the inductive argument, we can restrict ourselves to considering only asymptotically holomorphic sections whose jet is γ-transverse to the strata S_k^b for $b < a$, for some fixed constant γ (this constant γ is half of the transversality estimate obtained in the previous step ; by assumption we only consider perturbations which are small enough to preserve γ-transversality to the previous strata). With this understood, say that a section s_k satisfies $\mathcal{P}_k(\eta, x)$ if either $j^r s_k(x)$ lies at distance more than η from S_k^a, or $j^r s_k(x)$ lies at distance less than $\frac{1}{4}\kappa\gamma - \eta$ from ∂S_k^a, or $j^r s_k$ is η-transverse to S_k^a at x. We want to show that the assumptions of Proposition 4.1 are satisfied by these properties.

Fix a point $x \in X$ and a constant $0 < \delta < \frac{1}{20}\kappa\gamma$, and consider asymptotically holomorphic sections s_k of E_k. First, if $j^r s_k(x)$ lies at distance less than $\frac{3}{20}\kappa\gamma$ from a point of $\partial S_k^a \cap S_k^b$ for some $b \prec a$, then the uniform bounds on covariant derivatives of s_k imply that the graph of $j^r s_k$ remains within distance less than $\frac{1}{5}\kappa\gamma$ of this point over a ball of fixed radius c_1 (independent of k, x or δ) around x. So, the property $\mathcal{P}_k(\frac{1}{20}\kappa\gamma, y)$ holds at every point $y \in B_{g_k}(x, c_1)$, and no perturbation is needed. In the rest of the argument, we can therefore assume that $j^r s_k(x)$ lies at distance at least $\frac{3}{20}\kappa\gamma$ from ∂S_k^a.

Let $\epsilon = \frac{1}{10}\kappa\gamma$, and let ρ_ϵ be the radius appearing in Definition 3.2. Without loss of generality we can assume that $\rho_\epsilon < \epsilon$. Assume that $j^r s_k(x)$ lies at distance more than $\frac{1}{2}\rho_\epsilon$ from S_k^a. Then, the bounds on covariant derivatives of s_k imply that the graph of $j^r s_k$ remains at distance more than $\frac{1}{4}\rho_\epsilon$ from S_k^a over a ball of fixed radius c_2 around x, and therefore that s_k satisfies $\mathcal{P}_k(\frac{1}{4}\rho_\epsilon, y)$ at every point $y \in B_{g_k}(x, c_2)$. No perturbation is needed.

Therefore, we may assume that $j^r s_k(x)$ lies at distance less than $\frac{1}{2}\rho_\epsilon$ from a certain point $u_0 \in S_k^a$. We may also safely assume that $\delta < \frac{1}{4}\rho_\epsilon$. One easily checks that u_0 lies at distance more than ϵ from ∂S_k^a. So we can find complex-valued functions $f_1, \dots, f_p$ over the ball $B_{\hat{g}_k}(u_0, \rho_\epsilon)$ such that a local equation of S_k^a is $f_1 = \cdots = f_p = 0$ and satisfying the various properties listed in Definition 3.2. Let c_3 be a fixed positive constant (independent of k, x and δ) such that the graph of $j^r s_k$ over $B_{g_k}(x, c_3)$ is contained in $B_{\hat{g}_k}(u_0, \frac{3}{4}\rho_\epsilon)$, and define the $\mathbb{C}^p$-valued function $h = (f_1 \circ j^r s_k, \dots, f_p \circ j^r s_k)$ over $B_{g_k}(x, c_3)$. By property (4) of Definition 3.2, the function h is C^2-approximately holomorphic.

Recall from Lemma 2.2 that there exist local approximately holomorphic ω_k-Darboux coordinates $z_1, \dots, z_n$ over a neighborhood of x in X. Also recall from Lemma 2.3 that there exist approximately holomorphic sections $s^{\text{ref}}_{k,x,i}$ of E_k with Gaussian decay away from x and defining a local frame in E_k. For any $(n+1)$-tuple $I = (i_0, i_1, \dots, i_n)$ with $1 \le i_0 \le m$, $i_1, \dots, i_n \ge 0$, and $i_1 + \cdots + i_n \le r$, we define $s^{\text{ref}}_{k,x,I} = z_1^{i_1} \dots z_n^{i_n} s^{\text{ref}}_{k,x,i_0}$. Clearly, these sections of E_k are asymptotically holomorphic and have uniform Gaussian decay away from x ; moreover it is easy to check that their r-jets define a local frame in $\mathcal{J}^r E_k$ near x. After multiplication by a suitable fixed constant factor, we can also assume that $|s^{\text{ref}}_{k,x,I}|_{C^{r+1}, g_k} \le \frac{1}{p}$. For each tuple I, define a $\mathbb{C}^p$-valued function Θ_I by $\Theta_I(y) = (df_1(j^r s_k(y)).j^r s^{\text{ref}}_{k,x,I}(y), \dots, df_p(j^r s_k(y)).j^r s^{\text{ref}}_{k,x,I}(y))$. The functions Θ_I measure the variations of the function h when small multiples of the localized perturbations $s^{\text{ref}}_{k,x,I}$ are added to s_k ; by condition (4) of Definition 3.2, they are C^2-asymptotically holomorphic.

The fact that the jets of $s^{\text{ref}}_{k,x,I}$ define a frame of $\mathcal{J}^r E_k$ near x implies, by condition (2) of Definition 3.2, that the values $\Theta_I(x)$ generate all of $\mathbb{C}^p$. Moreover, for $1 \le i \le p$ there exist complex constants $\lambda_{I,i}$ with $\sum_I |\lambda_{I,i}| \le 1$ such that, defining the linear combinations $\sigma_{k,x,i} = \sum_I \lambda_{I,i} s^{\text{ref}}_{k,x,I}$ and $\Theta_i = \sum_I \lambda_{I,i} \Theta_I$, the quantity $|\Theta_1(x) \wedge \cdots \wedge \Theta_p(x)|$ is larger than some fixed positive constant $\beta > 0$ depending only on ϵ (and not on k, x or δ). The uniform bounds on derivatives imply that, for some fixed constant $0 < c_4 < c_3$, the norm of $\Theta_1 \wedge \cdots \wedge \Theta_p$ remains larger than $\frac{1}{2}\beta$ at every point of $B_{g_k}(x, c_4)$. Therefore, over this ball we can express h in the form $h = \mu_1 \Theta_1 + \cdots + \mu_p \Theta_p$, and the $\mathbb{C}^p$-valued function $\mu = (\mu_1, \dots, \mu_p)$ is easily checked to be C^2-AH as well.

Finally, use once more the local approximately holomorphic coordinates to identify $B_{g_k}(x, c_4)$ with a neighborhood of the origin in $\mathbb{C}^n$. After rescaling the coordinates by a fixed constant factor, we can assume that this neighborhood contains the ball B^+ of radius $\frac{11}{10}$ around the origin in $\mathbb{C}^n$, and that there exists a fixed constant $0 < c_5 < c_4$ such that the inverse image of the unit ball B in $\mathbb{C}^n$ contains $B_{g_k}(x, c_5)$. Composing μ with the coordinate map, we obtain a $\mathbb{C}^p$-valued function $\tilde{\mu}$ over B^+ ; by construction $\tilde{\mu}$ is C^2-AH.

We may now use the following local result, due to Donaldson [8] (the case $p = 1$ is an earlier result of Donaldson [7]; the comparatively much easier case $p > n$ is handled in [3]) :

Proposition 4.2 (Donaldson [8])**.** *Let f be a function with values in $\mathbb{C}^p$ defined over the ball B^+ of radius $\frac{11}{10}$ in $\mathbb{C}^n$. Let δ be a constant with $0 < \delta < \frac{1}{2}$, and let $\eta = \delta \log(\delta^{-1})^{-\nu}$ where ν is a suitable fixed integer depending only on n and p. Assume that f satisfies the following bounds over B^+:*

$$|f| \le 1, \qquad |\bar\partial f| \le \eta, \qquad |\nabla \bar\partial f| \le \eta.$$

Then, there exists $w \in \mathbb{C}^p$, with $|w| \le \delta$, such that $f - w$ is η-transverse to 0 over the interior ball B of radius 1.

Let $\eta = \delta \log(\delta^{-1})^{-\nu}$ as in the statement of the proposition, and observe that, if k is large enough, the antiholomorphic derives of $\tilde\mu$, which are bounded by a fixed multiple of $c_k^{-1/2}$, are smaller than η. Therefore, if k is large enough we can apply Proposition 4.2 (after a suitable rescaling to ensure that $\tilde\mu$ is bouded by 1) and find a constant $w = (w_1, \dots, w_p) \in \mathbb{C}^p$, smaller than δ, such that $\tilde\mu - w$ is η-transverse to 0 over the unit ball B. Going back through the coordinate map, this implies that $\mu - w$ is $c'_1\eta$-transverse to 0 over $B_{g_k}(x, c_5)$ for some fixed constant c'_1. Multiplying by the functions $\Theta_1, \dots, \Theta_p$, we obtain that $h - (w_1\Theta_1 + \dots + w_p\Theta_p)$ is $c'_2\eta$-transverse to 0 over $B_{g_k}(x, c_5)$ for some fixed constant c'_2.

Let $\tau_{k,x} = -(w_1\sigma_{k,x,1} + \dots + w_p\sigma_{k,x,p})$: by construction, the sections $\tau_{k,x}$ of E_k are asymptotically holomorphic, their norm is bounded by δ, and they have uniform Gaussian decay properties. Let $\tilde s_k = s_k + \tau_{k,x}$, and observe that by construction the graph of $j^r\tilde s_k$ over $B_{g_k}(x, c_3)$ is contained in $B_{\hat g_k}(u_0, \rho_\epsilon)$. Define $\tilde h = (f_1 \circ j^r\tilde s_k, \dots, f_p \circ j^r\tilde s_k)$; by construction, and because of the bounds on second derivatives of $f_1, \dots, f_p$, we have the equality $\tilde h = h - (w_1\Theta_1 + \dots + w_p\Theta_p) + O(\delta^2)$. If δ is assumed to be small enough, the quadratic term in this expression is much smaller than η ; therefore, under this assumption we get that $\tilde h$ is $c'_3\eta$-transverse to 0 over $B_{g_k}(x, c_5)$ for some fixed constant c'_3. Finally, recalling the characterization of estimated transversality to a submanifold defined by local equations given at the end of §3.1, we conclude that the graph of $j^r\tilde s_k$ is $c'_4\eta$-transverse to S_k^a over $B_{g_k}(x, c_5)$ for some fixed constant c'_4, i.e. $\tilde s_k$ satisfies the property $\mathcal{P}_k(c'_4\delta\log(\delta^{-1})^{-\nu}, y)$ at every point $y \in B_{g_k}(x, c_5)$.

Putting together the various possible cases (according to the distance between $j^r s_k(x)$ and S_k^a or its boundary), we obtain that the properties $\mathcal{P}_k$ satisfy the assumptions of Proposition 4.1. Therefore, for all large values of k a small perturbation can be added to s_k in order to achieve uniform transversality to S_k^a away from ∂S_k^a. The inductive argument described at the beginning of this section then makes it possible to complete the proof of Theorem 1.1.

The proof of Theorem 3.2 follows the same argument, but for one-parameter families of sections. One easily checks that the various results of §2 (Lemma 2.1, 2.2, 2.3) remain valid for families of objects depending continuously on a parameter $t \in [0, 1]$. Moreover, Propositions 4.1 and 4.2 also extend to the one-parameter case [8, 3]. So we only need to check that the argument used above to verify that the properties $\mathcal{P}_k$ satisfy the assumptions of Proposition 4.1 extends to the case of one-parameter families.

As before, fix a stratum $S^a_{k,t}$ in each stratification, a constant $\delta > 0$, a point $x \in X$, and asymptotically holomorphic sections $s_{k,t}$ of $E_{k,t}$. With the same notations as above, let $\Omega_k \subset [0,1]$ be the set of values of t such that $j^r s_{k,t}(x)$ lies at distance more than $\frac{3}{20}\kappa\gamma$ from $\partial S^a_{k,t}$, and within distance $\frac{1}{2}\rho_\epsilon$ from $S^a_{k,t}$. Let $\Omega^-_k \subset \Omega_k$ be the set of values of t such that $j^r s_{k,t}(x)$ lies at distance more than $\frac{1}{5}\kappa\gamma$ from $\partial S^a_{k,t}$ and within distance $\frac{1}{4}\rho_\epsilon$ from $S^a_{k,t}$. Observe that, if $t \notin \Omega^-_k$, a certain uniform transversality property with respect to $S^a_{k,t}$ is already satisfied by $j^r s_{k,t}$ over a small ball centered at x, and therefore no specific perturbation is needed : if x lies within distance $\frac{1}{5}\kappa\gamma$ from $\partial S^a_{k,t}$, then $\mathcal{P}_k(\frac{1}{40}\kappa\gamma, y)$ is satisfied at every point of a ball of fixed radius, while if x lies at distance more than $\frac{1}{4}\rho_\epsilon$ from $S^a_{k,t}$ then $\mathcal{P}_k(\frac{1}{8}\rho_\epsilon, y)$ holds over a ball of fixed radius around x. Even better, if δ is small enough compared to γ and ρ_ϵ, then any perturbation of $s_{k,t}$ by less than δ still satisfies a similar transversality property (with decreased estimates).

For t in Ω_k, the proximity of $j^r s_{k,t}(x)$ to $S^a_{k,t}$ makes it possible to locally define complex-valued functions $f_{1,t}, \dots, f_{p,t}$ depending continuously on t and such that a local equation of $S^a_{k,t}$ is $f_{1,t} = \dots = f_{p,t} = 0$ (recall the definition of the continuous dependence of the stratifications $\mathcal{S}_{k,t}$ upon the parameter t given in §3.2). This lets us define as above the function $h_t = (f_{1,t} \circ j^r s_{k,t}, \dots, f_{p,t} \circ j^r s_{k,t})$, depending continuously on t. As in the non-parametric case, we can construct asymptotically holomorphic sections $s^{\text{ref}}_{k,x,t,I}$ of $E_{k,t}$, with Gaussian decay away from x and defining local frames in $\mathcal{J}^r E_{k,t}$, simply by multiplying the sections of Lemma 2.3 by polynomials of degree at most r in the local coordinates (all these sections depend continuously on t). We can then find linear combinations $\sigma_{k,x,t,1}, \dots, \sigma_{k,x,t,p}$ of the sections $s^{\text{ref}}_{k,x,t,I}$, with constant coefficients depending continuously on t, such that, denoting by $\Theta_{t,i}$ the $\mathbb{C}^p$-valued functions expressing the variations of h_t upon adding small multiples of $\sigma_{k,x,t,i}$ to $s_{k,t}$, the norm of $\Theta_{t,1} \wedge \dots \wedge \Theta_{t,p}$ is bounded from below at x and over a small ball surrounding it.

Constructing the functions $\tilde{\mu}_t$ as in the proof of Theorem 1.1 and applying the one-parameter version of Proposition 4.2, we obtain, if k is large enough, a continuous one-parameter family of constants $w_t \in \mathbb{C}^p$, depending continuously on $t \in \Omega_k$ and bounded by δ for all t, such that $\tilde{\mu}_t - w_t$ is η-transverse to 0 over the unit ball in $\mathbb{C}^n$. It follows that, denoting by $\tau_{k,x,t}$ the asymptotically holomorphic perturbations $-(w_{t,1}\sigma_{k,x,t,1} + \dots + w_{t,p}\sigma_{k,x,t,p})$, bounded by δ, with Gaussian decay away from x, and depending continuously on $t \in \Omega_k$, the sections $s_{k,t} + \tau_{k,x,t}$ satisfy the desired transversality property over a small ball centered at x. However these perturbations are only well-defined for $t \in \Omega_k$. In order to extend their definition to all values of t, let $\chi_k : [0,1] \to [0,1]$ be a continuous cut-off function such that $\chi_k(t) = 1$ for all $t \in \Omega^-_k$ and $\chi_k(t) = 0$ for all $t \notin \Omega_k$, and let $\tilde{\tau}_{k,x,t} = \chi_k(t)\tau_{k,x,t}$ (for $t \notin \Omega_k$ we set $\tilde{\tau}_{k,x,t} = 0$). For $t \in \Omega^-_k$ we have $\tilde{\tau}_{k,x,t} = \tau_{k,x,t}$, so the sections $s_{k,t} + \tilde{\tau}_{k,x,t}$ satisfy the required transversality property ; for $t \notin \Omega^-_k$, the sections $s_{k,t}$ already satisfy such a property and, because we have assumed δ to be small enough, transversality is not affected by adding $\tilde{\tau}_{k,x,t}$.

Therefore, the assumptions of Proposition 4.1 are satisfied even in the one-parameter setting, and we can conclude the argument in the same way as in the non-parametric case.

Remark. In many cases, Theorems 1.1 and 3.2 can be proved without using Proposition 4.2 (estimated Sard lemma) in its full generality. Indeed, given suitable asymptotically holomorphic quasi-stratifications $\mathcal{S}_k$ of $\mathcal{J}^r E_k$, we can define quasi-stratifications $\tilde{\mathcal{S}}_k$ of $\mathcal{J}^{r+1}E_k$ in the following way. View each element of $\mathcal{J}^{r+1}E_k$ as a 1-jet in $\mathcal{J}^r E_k$, as in §3.2; for each stratum S_k^a of $\mathcal{S}_k$ with codimension greater than n in $\mathcal{J}^r E_k$, let $\tilde{S}_k^a$ be the set of points in $\mathcal{J}^{r+1}E_k$ whose r-jet component belongs to S_k^a. For each stratum S_k^a of $\mathcal{S}_k$ with codimension $p \leq n$, and for each value $0 \leq i \leq p-1$, let $\tilde{S}_k^{a,i}$ be the set of points in $\mathcal{J}^{r+1}E_k$ whose r-jet component belongs to S_k^a and such that the corresponding element in $T^*X^{(1,0)} \otimes \mathcal{J}^r E_k$, after projection to the normal space to TS_k^a, has rank equal to i. In other terms, the union of $\tilde{S}_k^{a,i}$ is the set of $(r+1)$-jets which intersect S_k^a non-transversely.

In a large number of examples, those of the $\tilde{S}_k^{a,i}$ which are not empty are approximately holomorphic submanifolds of $\mathcal{J}^{r+1}E_k$, transverse to the fibers and of codimension at least $n+1$. These submanifolds determine finite Whitney quasi-stratifications $\tilde{\mathcal{S}}_k$ of $\mathcal{J}^{r+1}E_k$, satisfying properties similar to those of Definition 3.2 but with C^1 estimates only instead of C^2 bounds. Still, the same argument as in the proof of Theorem 1.1 shows that, given asymptotically holomorphic sections s_k of E_k, small perturbations can be added for large enough k in order to ensure the uniform transversality of $j^{r+1}s_k$ to $\tilde{\mathcal{S}}_k$; the argument only uses Proposition 4.2 in the case $p > n$, where the proof becomes much easier [3] and C^1 bounds are sufficient. Because all the strata are of codimension greater than n, the η-transversality of $j^{r+1}s_k$ to $\tilde{\mathcal{S}}_k$ simply means that the graph of $j^{r+1}s_k$ remains at distance more than η from the strata of $\tilde{\mathcal{S}}_k$. By definition of $\tilde{\mathcal{S}}_k$, this is equivalent to the uniform transversality of $j^r s_k$ to $\mathcal{S}_k$, which was the desired result.

5. Examples and applications

We now consider various examples of (quasi)-stratifications to which we can apply Theorems 1.1 and 3.2. The fact that they are asymptotically holomorphic is in all cases a direct consequence of Proposition 3.1.

To make things more topological, we place ourselves in the case where the almost-complex structure J on X is tamed by a given symplectic form ω. In this context, the various approximately J-holomorphic submanifolds of X appearing in the constructions are automatically symplectic with respect to ω. Moreover, remember that the space of ω-tame or ω-compatible almost-complex structures on X is contractible. In most applications, asymptotically very ample bundles are constructed from line bundles with first Chern class proportional to $[\omega]$; in that situation, the ampleness properties of these bundles do not depend on the choice of an ω-compatible almost-complex structure J. Theorem 3.2 then implies that all the constructions described below are, for large enough values of k, canonical up to isotopy, independently of the choice of J. In the general case, the constructions are still canonical up to isotopy, but the space of possible choices for J is constrained by the necessity for the bundles E_k to be ample.

The first application is the construction of symplectic submanifolds as zero sets of asymptotically holomorphic sections of vector bundles over X, as initially obtained by Donaldson [7] and later extended to a slightly more general setting [2].

Corollary 5.1. *Let (X,ω) be a compact symplectic manifold endowed with an ω-tame almost-complex structure J, and let E_k be an asymptotically very ample sequence of locally splittable vector bundles over (X,J). Then, for all large enough values of k there exist asymptotically holomorphic sections s_k of E_k which are uniformly transverse to 0 and whose zero sets are smooth symplectic manifolds in X. Moreover these sections and submanifolds are, for large k, canonical up to isotopy, indepedently of the chosen almost-complex structure on X.*

Proof. Let $\mathcal{S}_k$ be the stratification of $\mathcal{J}^0 E_k = E_k$ in which the only stratum is the zero section of E_k (these stratifications are obviously asymptotically holomorphic). By Theorem 1.1, starting from any asymptotically holomorphic sections of E_k (e.g. the zero sections) we can obtain for large k asymptotically holomorphic sections of E_k which are uniformly transverse to $\mathcal{S}_k$, i.e. uniformly transverse to 0. It is then a simple observation that the zero sets of these sections are, for large k, smooth approximately J-holomorphic (and therefore symplectic) submanifolds of X [7]. Finally, the uniqueness of the construction up to isotopy is a direct consequence of the one-parameter result Theorem 3.2 [2]. □

The next example is that of determinantal submanifolds as constructed by Muñoz, Presas and Sols [11].

Corollary 5.2. *Let (X,ω) be a compact symplectic manifold endowed with an ω-tame almost-complex structure J, let L_k be an asymptotically very ample sequence of line bundles over (X,J), and let E and F be complex vector bundles over X. Then, for all large enough values of k there exist asymptotically holomorphic sections s_k of $E^*\otimes F\otimes L_k$ such that the determinantal loci $\Sigma_i(s_k)=\{x\in X,\ \mathrm{rk}(s_k(x))=i\}$ are stratified symplectic submanifolds in X. Moreover these sections and submanifolds are, for large k, canonical up to isotopy, indepedently of the chosen almost-complex structure on X.*

Proof. Let $E_k=E^*\otimes F\otimes L_k$, and let $\mathcal{S}_k$ be the stratification of $\mathcal{J}^0 E_k = E_k$ consisting of strata S_k^i, $0\le i<\min(\mathrm{rk}\,E,\mathrm{rk}\,F)$, defined as follows : viewing the points of E_k as elements of $\mathrm{Hom}(E,F)$ with coefficients in L_k, each S_k^i is the set of all elements in E_k whose rank is equal to i. By Proposition 3.1, the stratifications $\mathcal{S}_k$ are asymptotically holomorphic. Applying Theorem 1.1 to these stratifications and starting from the zero sections, we obtain asymptotically holomorphic sections of E_k which are uniformly transverse to $\mathcal{S}_k$. The determinantal locus $\Sigma_i(s_k)$ is precisely the set of points where the graph of s_k intersects the stratum S_k^i. The result of uniqueness up to isotopy is obtained by applying Theorem 3.2. □

However, our main application is that of maps to projective spaces. Observe that, given a section $s=(s_1,\dots,s_{m+1})$ of a vector bundle of the form

$\mathbb{C}^{m+1} \otimes L$, where L is a line bundle over X, we can construct away from its zero set a projective map $\mathbb{P}s = (s_1 : \ldots : s_{m+1}) : X - s^{-1}(0) \to \mathbb{CP}^m$.

Recall that the space of jets of holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^m$ carries a natural partition into submanifolds, the Boardman "stratification" [1, 6]. Restricting oneself to *generic* r-jets, the strata Σ_I, labelled by r-tuples $I = (i_1, \ldots, i_r)$ with $i_1 \geq \cdots \geq i_r \geq 0$, are defined in the following way. Given a generic holomorphic map f, call $\Sigma_i(f)$ the set of points where $\dim \operatorname{Ker} df = i$, and denote by Σ_i the set of holomorphic 1-jets corresponding to such points (i.e., Σ_i is the set of 1-jets (σ_0, σ_1) such that $\dim \operatorname{Ker} \sigma_1 = i$). The submanifolds Σ_i determine a stratification of $\mathcal{J}^1_{n,m}$. For a generic holomorphic map f the critical loci $\Sigma_I(f)$ are smooth submanifolds defining a partition of $\mathbb{C}^n$. Therefore, we can define inductively $\Sigma_{i_1,\ldots,i_r}(f)$ as the set of points of $\Sigma_{i_1,\ldots,i_{r-1}}(f)$ where the kernel of the restriction of df to $T\Sigma_{i_1,\ldots,i_{r-1}}(f)$ has dimension i_r (in particular, $\Sigma_{i_1,\ldots,i_{r-1},0}(f)$ is open in $\Sigma_{i_1,\ldots,i_{r-1}}(f)$ and corresponds to the set of points where f restricts to $\Sigma_{i_1,\ldots,i_{r-1}}(f)$ as an immersion).

It is easy to check that the r-jet of f at a given point of $\mathbb{C}^n$ completely determines in which $\Sigma_I(f)$ it lies ; therefore, one can define $\Sigma_I \subset \mathcal{J}^r_{n,m}$ as the set of r-jets $j^r f(x)$ of generic holomorphic maps $f : \mathbb{C}^n \to \mathbb{C}^m$ at points $x \in \Sigma_I(f)$. In other terms, $\Sigma_I(f) = \{x \in \mathbb{C}^n,\ j^r f(x) \in \Sigma_I\}$. It is a classical result [6] that the Σ_I's are smooth submanifolds and define a partition of the space of generic holomorphic r-jets (an open subset in $\mathcal{J}^r_{n,m}$ whose complement has codimension $\geq n+1$), which can be extended into a partition of $\mathcal{J}^r_{n,m}$ by smooth submanifolds.

The Boardman classes Σ_I play a fundamental role in singularity theory, and they completely determine the classification of singularities in certain dimensions. For low enough values of r, m or n, the submanifolds Σ_I define a genuine stratification of the jet space $\mathcal{J}^r_{n,m}$. However, as observed by Boardman, things become more complicated as the dimension increases, and the boundary of Σ_I is in general not a union of entire strata ; in high dimensions Boardman classes do not even define a quasi-stratification.

Still, there exist well-known methods that allow Boardman's partitions to be refined into finite Whitney stratifications of $\mathcal{J}^r_{n,m}$. An example of such a construction can be found in the work of Mather [10] (the constructed object is tautologically a finite Whitney stratification, and one easy checks that each Boardman class is a union of several of its strata).

We now consider the case of maps to projective spaces defined by asymptotically holomorphic sections of $E_k = \mathbb{C}^{m+1} \otimes L_k$ over X. We want to construct a natural approximately holomorphic analogue of the Thom-Boardman stratifications, by defining certain submanifolds in $\mathcal{J}^r E_k$. In order to make things easier by avoiding a lengthy analysis of the boundary structure at the points where the vanishing of the section prevents the definition of a projective map, our aim will only be to construct quasi-stratifications of $\mathcal{J}^r E_k$ rather than genuine stratifications.

We first define $Z = \{(\sigma_0, \ldots, \sigma_r) \in \mathcal{J}^r E_k,\ \sigma_0 = 0\}$, i.e. Z is the set of r-jets of sections which vanish at the considered point. As observed in §3.2, Θ_Z consists of all points of Z such that σ_1 is surjective. Next, observe that any point $(\sigma_0, \ldots, \sigma_r) \in \mathcal{J}^r E_k$ which does not belong to Z determines the (symmetric) holomorphic r-jet $(\phi_0,\dots,\phi_r)$ of a map to $\mathbb{CP}^m$: $\phi_0 \in \mathbb{CP}^m$, $\phi_1 \in T^*_x X^{1,0} \otimes T_{\phi_0}\mathbb{CP}^m$, ..., $\phi_r \in (T^*_x X^{1,0})^{\otimes r}_{\mathrm{sym}} \otimes T_{\phi_0}\mathbb{CP}^m$ are defined in terms of $\sigma_0,\dots,\sigma_r$ by expressions involving the projection map from $\mathbb{C}^{m+1}-\{0\}$ to $\mathbb{CP}^m$ and its derivatives. In fact, one easily checks that, if $(\sigma_0,\dots,\sigma_r) = j^r s$ is the symmetric holomorphic part of the r-jet of a section of E_k, then $(\phi_0,\dots,\phi_r) = j^r f$ is the symmetric holomorphic part of the r-jet of the corresponding projective map. Using this notation, define

$$\Sigma_i = \{(\sigma_0,\dots,\sigma_r) \in \mathcal{J}^r E_k,\ \sigma_0 \neq 0,\ \dim \operatorname{Ker} \phi_1 = i\}.$$

For $\max(0, n-m) < i \le n$, one easily checks that Σ_i is a smooth submanifold of $\mathcal{J}^r E_k$, and that $\partial\Sigma_i$ is the union of $\bigcup_{j>i} \Sigma_j$ and a subset of $Z - \Theta_Z$: indeed, observe that if $n \ge m$, then for any $(\sigma_0,\dots,\sigma_r) \in \overline{\Sigma}_i \cap Z$ we have $\dim \operatorname{Ker} \sigma_1 \ge i-1 > n-(m+1)$ and therefore σ_1 is not surjective, while in the case $n < m$ dimensional reasons prevent σ_1 from being surjective.

Next, we assume that $r \ge 2$, and observe that Θ_{Σ_i} is the set of points $(\sigma_0,\dots,\sigma_r) \in \Sigma_i$ such that

$$\Xi_{i;(\sigma_0,\dots,\sigma_r)} = \{u \in T_x X^{1,0},\ (\iota_u\sigma_1,\dots,\iota_u\sigma_r,0) \in T_{(\sigma_0,\dots,\sigma_r)}\Sigma_i\}$$

has the expected codimension in $T_x X^{1,0}$ (i.e., the same codimension as Σ_i in $\mathcal{J}^r E_k$). Indeed, by definition $(\sigma_0,\dots,\sigma_r)$ belongs to Θ_{Σ_i} if and only if the $(r+1)$-jet $(\sigma_0,\dots,\sigma_r,0)$, viewed as a 1-jet in $\mathcal{J}^r E_k$, intersects Σ_i transversly (because the definition of Σ_i involves only σ_0 and σ_1, the choice of a lift in $\mathcal{J}^{r+1}E_k$ does not matter, so we can choose the $(r+1)$-tensor component to be zero). By convention (see §3.2), this element of $\mathcal{J}^{r+1}E_k$ corresponds to the 1-jet of a local section σ of $\mathcal{J}^r E_k$ satisfying, at the given point $x \in X$, $\sigma(x) = (\sigma_0,\dots,\sigma_r)$ and $\nabla\sigma(x) = (\sigma_1,\dots,\sigma_{r+1})$: the covariant derivative contains no antiholomorphic or antisymmetric terms. The graph of σ intersects Σ_i transversely if and only if $\{u \in TX,\ \nabla\sigma(x).u \in T_{\sigma(x)}\Sigma_i\}$ has the expected dimension, hence the above criterion.

With this understood, we can define inductively, for $p+1 \le r$,

$$\Sigma_{i_1,\dots,i_{p+1}} = \{\sigma \in \Theta_{\Sigma_{i_1,\dots,i_p}},\ \dim(\operatorname{Ker}\phi_1 \cap \Xi_{(i_1,\dots,i_p);\sigma}) = i_{p+1}\},$$

where $\Xi_{I;\sigma} = \{u \in T_x X^{1,0},\ (\iota_u\sigma_1,\dots,\iota_u\sigma_r,0) \in T_{(\sigma_0,\dots,\sigma_r)}\Sigma_I\}$ as above, and Θ_{Σ_I} again consists of all points $\sigma \in \Sigma_I$ such that $\Xi_{I;\sigma}$ has the same codimension in $T_x X^{1,0}$ as Σ_I in $\mathcal{J}^r E_k$.

For $i_1 \ge \cdots \ge i_{p+1} \ge 1$, $\Sigma_{i_1,\dots,i_{p+1}}$ is a smooth submanifold in $\mathcal{J}^r E_k$, and its closure inside $\Sigma_{i_1,\dots,i_p}$ is obtained by adding $\bigcup_{j>i_{p+1}} \Sigma_{i_1,\dots,i_p,j}$ and a subset of $\Sigma_{i_1,\dots,i_p} - \Theta_{\Sigma_{i_1,\dots,i_p}}$. However, it is quite difficult to fully understand the boundary structure of $\Sigma_{i_1,\dots,i_{p+1}}$; the situation is exactly the same as in standard Boardman theory for holomorphic jets, except that, besides pieces of $\Sigma_{j_1,\dots,j_q}$ where $q \le p+1$ and $(j_1,\dots,j_q) \ge (i_1,\dots,i_q)$ for the lexicographic order, the boundary of $\Sigma_{i_1,\dots,i_{p+1}}$ also contains a subset of $Z - \Theta_Z$.

In low dimensions and/or for low values of r, it can be checked that the submanifolds $Z, \Sigma_i, \Sigma_{i_1,i_2},\dots,\Sigma_{i_1,\dots,i_r}$ determine a finite Whitney quasi-stratification of $\mathcal{J}^r E_k$; for example when $r=1$ this is an immediate consequence of the above discussion.

However, in larger dimensions it is necessary to refine Boardman's construction as in the holomorphic case. The important observation is that, when $\mathcal{J}^r E_k$ is trivialized by choosing local asymptotically holomorphic coordinates and sections, the partition of $\mathcal{J}^r E_k - Z$ described above corresponds exactly to the partition of the space of r-jets of maps to $\mathbb{CP}^m$ given by Boardman classes. Therefore, we can circumvent the problem by refining Boardman's partition of $\mathcal{J}^r_{n,m}$ into a genuine stratification as explained above, lifting it by the projectivization map to a stratification of the space of non-vanishing jets in $\mathcal{J}^r_{n,m+1}$, and finally pull it back to obtain a stratification of $\mathcal{J}^r E_k - Z$. As in the holomorphic case, the Σ_I classes are realized as unions of strata ; therefore, transversality to this stratification implies transversality to the Σ_I's. Moreover, all strata (except for the open one which we discard anyway) are contained in the closure of Σ_1, so that the boundary structures near Z are entirely contained in $Z - \Theta_Z$; therefore adding Z to this stratification yields a quasi-stratification of $\mathcal{J}^r E_k$.

Definition 5.1. *Given asymptotically very ample line bundles L_k over the manifold (X^{2n}, J), and setting $E_k = \mathbb{C}^{m+1} \otimes L_k$, the* Boardman stratification *of $\mathcal{J}^r E_k$ is the quasi-stratification given by the submanifold Z and by a refined Thom-Boardman stratification of $\mathcal{J}^r E_k - Z$.*

Corollary 5.3. *Let (X, ω) be a compact symplectic manifold endowed with an ω-tame almost-complex structure J, let L_k be an asymptotically very ample sequence of line bundles over (X, J), and let $E_k = \mathbb{C}^{m+1} \otimes L_k$. Then, for all large enough values of k there exist asymptotically holomorphic sections s_k of E_k such that the r-jets $j^r s_k$ are uniformly transverse to the Boardman stratifications of $\mathcal{J}^r E_k$.*

In particular, the zero sets $Z_k = s_k^{-1}(0)$ are smooth symplectic codimension $2m$ submanifolds in X, and the holomorphic r-jets of the projective maps $f_k = \mathbb{P}s_k : X - Z_k \to \mathbb{CP}^m$ behave at every point in a manner similar to those of generic holomorphic maps from a complex n-fold to $\mathbb{CP}^m$. Moreover, the singular loci $\Sigma_I(f_k) = \{x \in X - Z_k,\ j^r f_k(x) \in \Sigma_I\}$ are smooth symplectic submanifolds of the expected codimension and define a partition of $X - Z_k$. Finally, the sections s_k and the maps f_k are, for large k, canonical up to isotopy, independently of the chosen almost-complex structure on X.

Proof. By construction the Boardman stratifications of $\mathcal{J}^r E_k$ satisfy the assumptions of Proposition 3.1, as in every fiber of $\mathcal{J}^r E_k$ they can be identified with the same holomorphic quasi-stratification of $\mathcal{J}^r_{n,m+1}$. As a consequence, they are asymptotically holomorphic, and the existence of asymptotically holomorphic sections of E_k with the desired transversality properties is an immediate consequence of Theorem 1.1. The properties of Z_k follow immediately from the uniform transversality to the stratum Z of vanishing sections, while the properties of f_k are direct consequences of the uniform transversality to the Boardman strata (recall that each Σ_I is smooth and is a union of strata). Finally, the uniqueness result is obtained by applying Theorem 3.2. □

Corollary 5.3 is, in a certain sense, a fundamental result of asymptotically holomorphic singularity theory. Still, it falls short of the natural goal that one may have in mind at this point, namely the construction of approximately holomorphic projective maps which are near every point of X

topologically conjugate in approximately holomorphic coordinates to generic holomorphic maps between complex manifolds.

Indeed, in order to achieve such a result, one needs to obtain some control on the antiholomorphic part of the jet of f_k at the points of the singular loci $\Sigma_I(f_k)$: roughly speaking, $\bar{\partial} f_k$ must be much smaller than ∂f_k in every direction and at every point, and when ∂f_k is singular this is no longer an immediate consequence of asymptotic holomorphicity and transversality. Note however that the behavior of f_k near the set of base points Z_k is always the expected one.

In many cases, it is possible to perturb slightly the sections s_k (by less than a fixed multiple of $c_k^{-1/2}$, which affects neither holomorphicity nor transversality properties) along the singular loci in order to obtain the proper topological picture for f_k.

The easiest case is $m \geq 2n$, where it is enough to consider 1-jets, and all the strata turn out to be of codimension greater than n ; the uniform transversality of s_k to the Boardman stratification then implies that the maps f_k are approximately holomorphic immersions. Moreover, when $m \geq 2n+1$ an arbitrarily small perturbation is enough to get rid of multiple points, thus giving approximately holomorphic embeddings into projective spaces, a result already obtained by Muñoz, Presas and Sols [11].

Next, we can consider the case $m = 1$, where 1-jets are again sufficient, and the only interesting Boardman stratum is Σ_n, of complex codimension n, corresponding to critical points of $\mathbb{CP}^1$-valued maps. The sections of $\mathbb{C}^2 \otimes L_k$ given by Corollary 5.3 vanish along smooth codimension 4 base loci ; moreover, the differential ∂f_k of the $\mathbb{CP}^1$-valued map f_k only vanishes at isolated points, and does so in a non-degenerate way. These transversality properties are precisely those imposed by Donaldson in his construction of symplectic Lefschetz pencils [8]; the only missing ingredient is an extra perturbation near the zeroes of ∂f_k in order to get rid of the antiholomorphic terms and therefore ensure that they are genuine non-degenerate critical points, thus making f_k a complex Morse function.

The last case we will consider is when $m = 2$. In this case, we need to consider 2-jets, and the relevant Boardman strata are Σ_{n-1}, of complex codimension $n-1$, and $\Sigma_{n-1,1}$, of complex codimension n (the other strata have codimension greater than n). The sections of $\mathbb{C}^3 \otimes L_k$ constructed by Corollary 5.3 vanish along smooth codimension 6 base loci. The $\mathbb{CP}^2$-valued maps f_k are submersions outside of the smooth symplectic curves $R_k = \Sigma_{n-1}(f_k)$, and the restriction of f_k to R_k is an immersion except at the points of $C_k = \Sigma_{n-1,1}(f_k)$. After a suitable perturbation in order to ensure the vanishing of some antiholomorphic derivatives of f_k along R_k, one obtains a situation similar to that described in previous papers [3, 4]: at every point of $R_k - C_k$, a local model for f_k in approximately holomorphic coordinates is $(z_1, \dots, z_n) \mapsto (z_1^2 + \dots + z_{n-1}^2, z_n)$, while at the points of C_k the local model becomes $(z_1, \dots, z_n) \mapsto (z_1^3 + z_1 z_n + z_2^2 + \dots + z_{n-1}^2, z_n)$ and the symplectic curve $f_k(R_k) \subset \mathbb{CP}^2$ presents an isolated cusp singularity.

In the general case, the most promising strategy to achieve topological conjugacy to generic holomorphic local models is to perturb the sections s_k in order to make sure that, along each stratum $\Sigma_I(f_k)$, the germ of f_k

is holomorphic along the normal directions to $\Sigma_I(f_k)$. Such perturbations should be relatively easy to construct by methods similar to those in the above-mentioned papers[3, 4], provided that one starts from the strata of lowest dimension. This approach will be developped in a forthcoming paper.

Finally, let us formulate some natural extensions of Corollary 5.3 to more general situations. First, we mention the case when the asymptotically very ample line bundles L_k are replaced by vector bundles of rank $\nu \geq 2$. In that case, and provided that $m \geq \nu$, the projective maps defined by sections s_k of $\mathbb{C}^{m+1} \otimes L_k$ are replaced by maps $\mathrm{Gr}(s_k)$ taking values in the Grassmannian $\mathrm{Gr}(\nu, m+1)$ of ν-planes in $\mathbb{C}^{m+1}$, defined at every point of X where the $m+1$ chosen sections generate the whole fiber of L_k. More precisely, at every such point there exist $m+1-\nu$ independent linear relations between the $m+1$ components $s_k^1, \dots, s_k^{m+1}$, and these $m+1-\nu$ linear equations in $m+1$ variables determine a ν-dimensional complex subspace $\mathrm{Gr}(s_k)$ in $\mathbb{C}^{m+1}$. By adapting Corollary 5.3 to this situation, it is for example possible to recover the Grassmannian embedding result of Muñoz, Presas and Sols [11].

Another direction in which Corollary 5.3 can be improved is by adding extra transversality requirements to the projective maps f_k. For example, given a stratified holomorphic submanifold $\mathcal{D} = (D_a)_{a\in A}$ in $\mathbb{CP}^m$, we can require the transversality of the map f_k to $\mathcal{D}$. Indeed, $\mathcal{D}$ induces a stratification $\tilde{\mathcal{D}}_k$ of $\mathcal{J}^r E_k$, in which each stratum consists of the jets $(\sigma_0, \dots, \sigma_r)$ such that $\mathbb{P}\sigma_0$ belongs to a certain stratum D_a of $\mathcal{D}$ (in fact, this stratification only involves the 0-jet part). Starting from sections s_k of E_k given by Corollary 5.3, we can apply Theorem 1.1 to the stratifications $\tilde{\mathcal{D}}_k$ (which one easily shows to to be asymptotically holomorphic by Proposition 3.1) ; this yields asymptotically holomorphic sections $\tilde{s}_k$ which are uniformly transverse to $\tilde{\mathcal{D}}_k$ but differ from s_k by an amount small enough to ensure that the transversality of the jets to the Boardman stratification is preserved. In this way, one obtains projective maps which have the same properties as in Corollary 5.3 and additionally are uniformly transverse to the stratified submanifold $\mathcal{D}$. This extends a result of Muñoz, Presas and Sols [11] where asymptotically holomorphic embeddings are made transverse to a given submanifold of $\mathbb{CP}^m$.

Another class of stratifications of $\mathcal{J}^r E_k$ that we can consider are those obtained from lower-dimensional Boardman stratifications by linear projections. Namely, fix $q < m$, and let $\pi : \mathbb{C}^{m+1} \to \mathbb{C}^{q+1}$ be a linear projection ; π induces maps $\tilde{\pi} : \mathcal{J}^r(\mathbb{C}^{m+1} \otimes L_k) \to \mathcal{J}^r(\mathbb{C}^{q+1} \otimes L_k)$, and the inverse images by $\tilde{\pi}$ of the Boardman stratifications of $\mathcal{J}^r(\mathbb{C}^{q+1} \otimes L_k)$ are asymptotically holomorphic quasi-stratifications of $\mathcal{J}^r(\mathbb{C}^{m+1} \otimes L_k)$. The transversality of $j^r s_k$ to these quasi-stratifications is equivalent to that of $j^r(\pi(s_k))$ to the Boardman stratifications ; denoting by $\bar{\pi}$ the map from $\mathbb{CP}^m$ to $\mathbb{CP}^q$ induced by π, this is also equivalent to the genericity of the holomorphic jets of the projective maps $\bar{\pi} \circ f_k$. Therefore, by applying Theorem 1.1 as in the previous example, we can obtain projective maps f_k with the same genericity properties as in Corollary 5.3 and such that the maps $\bar{\pi} \circ f_k$ also enjoy similar properties. Even better, by iteratedly applying Theorem 1.1 we can obtain the same property for any given finite family of linear projections.

For example, when $m = 2$ and considering projections of $\mathbb{C}^3$ to $\mathbb{C}^2$ along coordinate axes, one obtains exactly the transversality properties which are needed in order to extend Moishezon-Teicher braid group techniques to the study of symplectic manifolds [5, 4].

To conclude, let us mention a different class of potential applications of Theorem 1.1, following the ideas of Donaldson and Smith. As shown by Donaldson [8], any compact symplectic 4-manifold carries structures of symplectic Lefschetz pencils obtained from pairs of sections of asymptotically very ample line bundles L_k ; after blowing up the base points, we obtain Lefschetz fibrations over $\mathbb{CP}^1$, which may also be thought of as maps from $\mathbb{CP}^1$ to the moduli space $\bar{M}_g$ of stable curves of a certain genus g. These maps become asymptotically holomorphic as one considers pencils given by sections of L_k for $k \to +\infty$. In a largely unexplored class of constructions, one considers certain vector bundles over $\mathbb{CP}^1$ naturally arising from the Lefschetz fibrations : for example, spaces of holomorphic sections of certain bundles over each fiber, or pull-backs by the maps from $\mathbb{CP}^1$ to $\bar{M}_g$ of vector bundles over $\bar{M}_g$. It often turns out that these bundles over $\mathbb{CP}^1$ either are naturally asymptotically very ample or become so after tensor product by the line bundles $O(k)$. Theorem 1.1 can then used in order to obtain sections with suitable genericity properties, which in turn give rise to interesting geometric or topological structures. In some cases the objects naturally arising are sheaves rather than bundles, but the same type of argument should remain valid. It is to be expected that some interesting results about symplectic 4-manifolds and Lefschetz pencils can be obtained in this way, as similar considerations (but at a much more sophisticated level) have for example led to Donaldson and Smith's proof of the existence of a pseudo-holomorphic curve realizing the canonical class via Lefschetz fibrations [9].

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Centre de Mathématiques, Ecole Polytechnique, F-91128 Palaiseau, France
E-mail address: auroux@math.polytechnique.fr